To Barbara Geary,

with many thanks for the typing

Lyn.

INITIATING A VERBAL SMALL-CANCELLATION THEORY

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DECLARATION

The work presented here is entirely my own.

Where it has been influenced by the work of others, this is explicitly stated and references are given.

Lambertus John Hesterman

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ABSTRACT

The main result proved in this thesis is presented in Chapter 3. It states that for arbitrary non-commuting words x, y in a free group there is a word z in the group, such that the cyclically reduced length of the commutator word

$$[x, y]\left[[x^2, y^2], [x, y]\right]^N \quad (N \text{ a positive integer})$$

is at least

$$4 + 8N + 4N\left(\lambda(x^z)+\lambda(y^z)\right) ,$$

where λ denotes length in the free group; the estimate is shown to be best-possible, in the sense that words x, y, z can be found for which the lower bound is attained, irrespective of lower bounds imposed on $\lambda(x)$, $\lambda(y)$ or upper bounds on $\lambda(z)$. The length-function used is the simplest one, in which a generator of the free group has length 1, and (for example) the square of a generator has length 2.

The method developed for proving this result seems to be new, and is applicable to a large class of words apart from the 2-variable commutator word shown here.

The result depends on a number of simpler ones, giving lower bounds for the lengths of various expressions in two or three variables under certain conditions. These subsidiary results are presented in Chapter 2. Also presented in Chapter 2 is a general method for finding and proving such results; this is based on what we call "cancellation flowcharts", which are used in tracing the progress of cancellation in an expression in several variables. (Our "cancellation flowcharts" are quite distinct from the "cancellation diagrams" of E.R. Van Kampen and R.C. Lyndon.)

Chapter 1 presents the basic machinery required for all this, and the Introduction discusses the relationship between this work and other contributions to combinatorial group theory.

INTRODUCTION

1. Presentations and small cancellation

Let $\langle X; R \rangle$ be a presentation of a group G. That is, let R be a set of words in the free group F generated by (the elements of) X, such that G is (isomorphic to) the quotient group F/N, where N is the normal closure of R in F. Two fundamental problems that one wishes to solve are, given two arbitrarily chosen words in F (that is, elements of F), to determine whether or not their images under the "quotient" homomorphism $F \to F/N$ are:

(i) equal, or
(ii) conjugate.

Succinct, non-trivial necessary and sufficient conditions giving the answers for all pairs of words are out of the question; what one seeks are algorithms that will give the answers for each given pair of words. These problems, known respectively as the "word problem" and the "conjugacy problem", are among three formulated by M. Dehn (1911); the third, which will not concern us, asks how one can tell whether two arbitrarily chosen group presentations are presentations of the same group (up to isomorphism). Clearly, the word problem is included in the conjugacy problem: an element of G conjugate to the identity element 1 of G is equal to 1, and an algorithm which can decide this solves the word problem.

Without restrictions on the class of presentations, the word problem is unsolvable, that is, every word-problem algorithm is defeated by some presentation. Indeed, there are presentations with finite X and R which defeat every algorithm (P.S. Novikov (1955), W.W. Boone (1955)).

A very strong restriction which makes the word problem solvable is that R contains just one relator. (W. Magnus (1932), Magnus, Karrass and Solitar (1966).)

Apart from this, the most successful way of restricting the class of presentations to be considered has been to impose "small-cancellation" conditions, to the effect that when elements of R are multiplied together in F relatively little cancellation can take place (except, of course, when an element of R is multiplied by its own inverse). First one replaces R by R^*, defined as the set comprising all cyclically reduced conjugates of elements of R and their inverses; clearly the normal closure of R^* in F coincides with the normal closure of R . Then one defines a "piece" to be any element u of F such that $r = ux$ and $r' = ux'$, where r and r' are distinct elements of R^* , the products being reduced without cancellation. The typical small-cancellation conditions (whose appellations have become standard in the literature) are now:

Condition $C'(\rho)$, where ρ is a positive real number (usually a simple fraction): If $r = ux$ (reduced without cancellation), where u is a piece and r is an element of R^* , then the length of u is less than ρ times the length of r .

Condition $C(p)$, where p is a positive integer: No element of R^* is a product of fewer than p pieces.

Condition $T(q)$, where q is a positive integer: If $r_1, r_2, \ldots, r_n$ are elements of R^* , $3 \leq n < q$, and none of the products $r_1 \cdot r_2, \ldots, r_{n-1} \cdot r_n$, $r_n \cdot r_1$ is the identity of F (the dots indicating possible cancellation), then at least one of these products is reduced without cancellation.

R.C. Lyndon (1966) solves the word problem for presentations satisfying $C(p)$ and $T(q)$, where (p, q) is (6, 3), (4, 4) or (3, 6) , and P.E. Schupp (1968, 1970 a) solves the conjugacy problem for the same class of presentations (the presentations are assumed to be finite: if the group G is not finitely presented then a word-problem algorithm is effective only if R^* is also completely specified by an algorithm, that is, if the presentation is recursive). These results improve on earlier work by V.A. Tartakovskii (1949 a, b, c) for condition $C(7)$, J.L. Britton (1956 a, b) and M. Greendlinger (1966) for $C'(\frac{1}{6})$, and H. Schiek (1956) for $C'(\frac{1}{4})$ and $T(4)$. Note that $C'(1/p)$ implies $C(p+1)$ for integral p , and that the condition $T(q)$ is vacuous for $q \leq 3$. The method developed by Lyndon and Schupp represents elements of F by certain labelled and oriented diagrams in the plane: cancellation then becomes a process of "glueing together" such diagrams in a specified way. (The basic idea was published by E.R. Van Kampen (1938), and independently rediscovered by Lyndon.) For a detailed and lucid exposition we refer the reader to the original papers, and to notes and a survey article by Schupp (1970 b). From this geometrical approach, as well as from Tartakovskii's work, it becomes evident that the really important role in such arguments is played by the "circular words" (which we shall define later, following Tartakovskii), which correspond in the geometrical approach to "free" oriented cycles, not connected to the origin.

2. The substitution problem

Suppose now, given the same sets X and R, that we take, not the normal closure N of R (invariant under all inner automorphisms of F), but the fully invariant closure V (mapped into itself by every endomorphism of F). The quotient group F/V is now a free group relative to a variety of groups, and R becomes a basis for the laws which define this variety. That is, each element of R is a word in the generators (elements of X), and the image of such a word under an endomorphism of F is obtained by substituting for each generator its image under that endomorphism, and then cancelling where possible. Thus the elements of X appearing in a word chosen from R act as variables; since every mapping from X into F extends to an endomorphism of F, the values of these variables are subject to no restriction.

Corresponding to the word problem, we have the problem of determining all the consequences of the set of laws R (the "consequences" are precisely the elements of V). If one neglects the requirement that the solution must be algorithmic, this is a paraphrase of the central problem of the theory of group varieties: classify all varieties of groups according to the laws which generate them, or, in other words, determine all the implications that hold between sets of laws. (For these topics, the canonical reference is Hanna Neumann (1967).) There are algorithms which solve this problem for certain sets R of laws: one such (for finite sets R containing a "nilpotency" law $[x_1, x_2, \ldots, x_n]$) is given in the author's earlier thesis (1970).

The analogue of the conjugacy problem is much less important: it asks for a method of determining whether two given words in F are in the same coset of V.

In addition to these, one may formulate a new problem, namely that of recognising a single "instance" of one of the laws in R. That is, given an arbitrary element h of F, we are to decide algorithmically whether or not there is an element r of R and an endomorphism η of F such that $h = r\eta$. The analogue of this for group presentations is the trivial "problem" of deciding whether or not a given element of F is in R. The importance of this recognition problem is clear from the following paraphrase: given an element h of F, and a set R of group words, to decide whether or not there is a group word $r(x_1, x_2, \ldots, x_n)$ contained in R, and elements $g_1, g_2, \ldots, g_n$ of F, such that substituting $g_1, \ldots, g_n$ for the variables $x_1, \ldots, x_n$ gives

$$r(g_1, g_2, \ldots, g_n) = h \ .$$

Clearly we may assume without loss of generality that R contains just one element. The problem therefore is, to solve a given homogeneous equation in F, or to ascertain that it has no solution.

Schupp (1969) has shown that this problem is solvable if the word r has two variables; the proof is very short, but depends on a rather notorious theorem by J.H.C. Whitehead (1936), whose proof is by no means short. More recently, a general procedure for solving homogeneous equations in two variables, avoiding Whitehead's theorem, has been given by M.J. Wicks (1972). Prior to this work only a few isolated results were known, the simplest and most elegant being the result by Wicks (1962), that an element of a free group is a commutator if and only if it has a cyclically reduced conjugate of the form $xyzx^{-1}y^{-1}z^{-1}$, where the product is reduced without cancellation. A more general result along the same lines was proved by B.B. Newman (1967), for words of the form $x^{-1}y^m xy^n$ (before cancellation). Newman shows that all such words, and only such words, have cyclically reduced conjugates taking one of a list of nine forms. Later, S. Meskin (1972) gave a somewhat shorter list, of four forms.

3. A "small-cancellation" method

A drawback to all of these methods of solving the substitution problem is that certain natural questions that one may wish to ask about the solutions of a homogeneous equation are answered only by solving the equation, and then studying the set of solutions.

The particular question we have in mind is: for a given word $r(x_1, x_2, \ldots, x_n)$ in the free group F, is there a lower bound on the length of r in terms of the lengths of the variables $x_1, x_2, \ldots, x_n$? More precisely, is it possible to find a formula $f(t_1, t_2, \ldots, t_n)$ such that $r(g_1, g_2, \ldots, g_n) = h$ implies

$$f\big(\lambda(g_1), \lambda(g_2), \ldots, \lambda(g_n)\big) \le \lambda(h)$$

where h is an arbitrary element of F? In general the answer is "no", unless we place some restrictions on the choice of the solution variables g_i, but if we stipulate that $(g_1, g_2, \ldots, g_n)$ is to be in some sense a minimal solution (for example, not conjugate to one of smaller total length), the answer may become both affirmative and interesting. Thus, if f is a strictly increasing function of each of the variables $t_i = \lambda(g_i)$, one has immediately an upper bound on the number of "minimal" solutions in terms of the length of h , and in particular one may be able to conclude that there are no non-trivial solutions unless the length of h is at least equal to some computed minimum.

In the present study we shall answer this question for a particular word r (more precisely, for a family of such words, parametrized by a positive integer variable), establishing a bound which, under the stated restrictions on the solution, is sharp. Like Schupp (1969) and Wicks (1972), we consider only words of two variables. In the course of our work, it will become evident that our method is quite general, and could easily be extended to give lower bounds of this kind for a very wide class of words. The bounds so computed would of course be different for different words r , so that no purpose would be served by an attempt to formulate a result in the widest generality, especially since the sharpness of the bound would thus be lost. The restriction to two variables is, I suspect, not particularly important, except in that an increase in the number of variables would require a more intricate statement of the restrictions to be placed on the solution-set, and a considerably more involved computation to arrive at the required bound.

The statement of our result places a lower bound on the cyclically reduced length of h , in terms of the lengths of certain conjugates of the variables appearing in r . The case in which the variables commute is easily handled and not very interesting, and in fact the word for which we shall find a lower bound is

$$r(x, y) = [x, y]\big[[x^2, y^2], [x, y]\big]^N ,$$

which vanishes if x, y commute. (Here N is a positive integer parameter.) We shall prove that if x, y do not commute, then there is an element z of F such that the cyclically reduced length of the above expression is at least $4 + 8N + 4N\big(\lambda(x^z)+\lambda(y^z)\big)$. The cyclically reduced length of the repeated portion, namely $\big[[x^2, y^2], [x, y]\big]$, is at least $8 + 4\big(\lambda(x^z)+\lambda(y^z)\big)$ for some z in F .

Our motivation for solving this particular problem is that, apart from serving as a good example for the method we have developed, it should serve also as a starting-point for the problem of showing that products of two or more instances must have length greater than some reasonably large value, if N is chosen suitably large. For our intended purpose, 4 is a "reasonably large value". The point is this: if no such product can have length 4 , then the verbal closure V of the word (that is, the law) $[x, y][[x^2, y^2], [x, y]]^N$ is properly contained in the commutator subgroup (first derived group) F' of F , and therefore the corresponding variety var(F/V) contains non-abelian groups. But in each group in this variety every commutator $[x, y]$ is in the second derived group, since

$$[x, y] = [[x^2, y^2], [x, y]]^{-N}$$

in all groups of var(F/V) . Thus, for every group G in the variety the first derived group G' is perfect (that is, $G' = G''$). The existence of such varieties has been a matter for conjecture.

A proof of such a lower bound for products of instances would justly be described as a "verbal small-cancellation" result. The method of proof presented in the following pages likewise centres on the problem of showing that in various expressions in several variables, culminating in the one we have quoted, when values are substituted for the variables the cancellations that ensue can remove no more than a certain portion of the total length of the reduced words appearing; it, too, may therefore be regarded as a part of "verbal small cancellation theory". Hence our title.

CHAPTER 1

1. The word semigroup M

Our reasoning will be conducted throughout in the context of a single free group F, generated by a set with an unspecified number of elements (at least two, otherwise the whole argument becomes trivial). As the notations and terminology that we shall find most convenient are not always the usual ones, we can best avoid ambiguity by introducing them in the course of an outline of the construction of F that we have in mind.

We begin with a non-empty set A, whose elements will be called *letters*, and a fixed-point-free involutory mapping defined on the set A. The image under this mapping of the arbitrary letter a will be denoted by $\bar{a}$. Thus, by definition, we have $\bar{\bar{a}} = a$ for each letter a (here, and throughout our work, the equality sign denotes identity). Obviously, the set A is partitioned by this mapping into two-element subsets $\{a, \bar{a}\}$. Letters belonging to different subsets in this partition will be called *independent*. Assuming enough of the Axiom of Choice to serve the purpose, we might now define a set X containing just one letter from every such pair - obviously X is destined to become the (or, rather, a) generating set for F - but in practice we can get along quite well without it.

Next, we form the set M, comprising all "words" obtainable by writing in sequence a finite number of (copies of) letters chosen from A. The number of letters so written is called the *length* of the word. If the word is denoted by w, then we denote by $\lambda(w)$ the length of w.

Included in M is the "empty word", namely the unique word of length 0, obtained by writing no letters at all. This word should be written as if we are to be perfectly consistent, but to avoid confusion we never *write* the empty word: instead we *denote* it by the symbol 1. (Here we follow mathematical custom: strictly speaking, this choice of symbol introduces an ambiguity between the word 1 and the integer 1, but the intended denotation will always be clear from context.)

As is natural, we define on M the binary operation of concatenation: given two words in M, denoted by w_1 and w_2 (say), we obtain their concatenation simply by juxtaposing them in this order, that is, by writing in sequence the letters of w_1, followed immediately by the letters of w_2, also in sequence. By definition, the resulting word is again an element of M. In denoting concatenation of words in M, we carry over the "juxtaposition" notation from words in M to any symbols used to denote them. So, if x denotes the word aba and y denotes the word $\bar{a}b\bar{b}$ (a and b being letters), then the word $aba\bar{a}b\bar{b}$ may be denoted by the expression xy. Clearly the set M with the binary operation of concatenation is a monoid, that is, a semigroup with the identity element 1, and by the usual arguments we may justify the use of expressions such as xyz or $w_1w_2 \ldots w_n$ (instead of fully bracketed forms such as $(xy)z$ or $x(yz)$) to denote words in M. As usual, a one-letter word whose only letter is a will be denoted by that letter (this justifies the use of "mixed" expressions such as axb, where a, b are letters and x denotes a word).

We make the usual convention regarding the "power" notation x^n for non-negative integers n : x^0 is defined as 1, and x^{n+1} is defined inductively as $x^n x$, for any word denoted by x. (What is perhaps less usual is that we shall stick to this convention even after we have defined F.)

If x denotes a word $a_1a_2 \ldots a_n$ in M, where $a_1, \ldots, a_n$ are (that is, denote) letters, not necessarily all different, then the symbol $\bar{x}$ may be used to denote the word $\bar{a}_n\bar{a}_{n-1} \ldots \bar{a}_1$, obtained by writing in reverse order the images of the letters of x under the involutory mapping $a \mapsto \bar{a}$. Clearly this notation is consistent with our previous one: if x denotes a one-letter word a then $\bar{x} = \bar{a}$. Likewise we have clearly

$$\bar{\bar{x}} = \bar{\bar{a}}_1\bar{\bar{a}}_2 \ldots \bar{\bar{a}}_n = a_1a_2 \ldots a_n = x .$$

If $x = 1$, then we make the convention $\bar{x} = 1$. We shall never use the "bar", or overline, notation over any expression comprising more than one symbol. (The reason is plain: if $z = xy$ then $\bar{z} = \bar{y}\bar{x}$, so that overlines of complex expressions could be confusing.)

For negative integers n, we interpret the expression x^n as denoting the word $(\bar{x})^{-n}$.

The procedure of concatenating two words x, y to give xy can obviously be reversed. For, the length of xy is just $\lambda(x) + \lambda(y)$: the first $\lambda(x)$ letters of xy, taken in sequence, comprise the word x, and the remaining $\lambda(y)$ letters, in sequence, comprise the word

y . So, given xy , we need only to know $\lambda(x)$ or $\lambda(y)$ to be able to recover (algorithmically) the words x and y .

Obviously, then, M is a cancellation semigroup: if $xy = xz$ then the first $\lambda(x)$ letters of xy comprise the word x , and the remaining $\lambda(y)$ $(= \lambda(z))$ letters comprise the word y and equally comprise the word z , so we have $y = z$. In the same way, $yx = zx$ implies $y = z$. More generally, we have the following.

THEOREM 1. *If x, y, u, v are words such that $xy = uv$, and if we have $\lambda(x) \geq \lambda(u)$ (or $\lambda(y) \leq \lambda(v)$), then there is a unique word w such that $x = uw$ and $v = wy$.*

Proof. The first $\lambda(x)$ letters of xy , comprising the word x , must include the first $\lambda(u)$ letters of uv $(= xy)$, which comprise the word u . Denoting by w the word formed by the remaining $\lambda(x) - \lambda(u)$ letters of x , we have then $x = uw$. Now $xy = uv$ gives $uwy = uv$, and hence $wy = v$, by the cancellation property. The uniqueness of w follows immediately from the cancellation property: if w_1 is another such word, then we have $x = uw = uw_1$, and hence $w = w_1$. //

COROLLARY 1.1. *If $\lambda(x) = \lambda(u)$ in the above, then clearly $\lambda(w) = 0$, hence $w = 1$, and we have $x = u$ and $y = v$.* //

COROLLARY 1.2. *If $xy = w^{n+1}$, where n is a non-negative integer, then there are words u, v and non-negative integers r, s such that $w = uv$, $r + s = n$, $x = w^r u$, and $y = vw^s$.*

Proof. The proof is trivial if $x = 1$ or $y = 1$: for $x = 1$, take $r = 0$, $u = 1$, and for $y = 1$ take $s = 0$, $v = 0$.

If $x \neq 1$ and $y \neq 1$, then clearly $w \neq 1$. By the Euclidean Algorithm, there is an integer $r \geq 0$ such that $r\lambda(w) \leq \lambda(x)$ and $(r+1)\lambda(w) > \lambda(x)$. By Theorem 1, there is a word u such that $x = w^r u$ and $uy = w^{s+1}$, where $s \geq 0$ is defined by $r + s = n$. Now $\lambda(x) = r\lambda(w) + \lambda(u) < (r+1)\lambda(w)$, hence $\lambda(u) < \lambda(w)$, so by Theorem 1 (and the equation $uy = w^{s+1}$) there is a word v such that $uv = w$ and $y = vw^s$. //

DEFINITION. If x, y, u, v are words such that $uxv = y$, then x is said to be a *subword* of y . If $u = 1$ we say that x is an *initial* subword of y , and if $v = 1$ we say that x is a *final* subword of y .

The "subword" relation so defined on M is a partial ordering:

reflexivity and transitivity are obvious, and the antisymmetry property follows by a length argument: $u_0xv_0 = y$ and $u_1yv_1 = x$ imply $u_1u_0xv_0v_1 = x$, and hence

$$\lambda(u_1) + \lambda(u_0) + \lambda(x) + \lambda(v_0) + \lambda(v_1) = \lambda(x) ,$$

so that u_1, u_0, v_0, v_1 are all of length 0 , and therefore all equal to 1 .

2. The free group F

DEFINITION. An *elementary word* is a word $a\bar{a}$, where a is any letter.

DEFINITION. The set F is defined as the set of all words x in M such that no elementary word is a subword of x . We shall refer to the elements of F as *reduced* words.

Because the "subword" relation is transitive, every subword of a reduced word is a reduced word. Thus, Theorem 1 and its Corollaries remain valid when "relativized" to F . That is, if the words x, y, u, v of Theorem 1 are in F then so is the word w (we need not insist that xy be in F). But the concatenation of two reduced words is not always reduced. That is, within F concatenation is not everywhere defined, but (of course) where it is defined it is associative.

Our next definition introduces "nonce" terms and notations, which will be used only within this Section.

DEFINITION. If x, y are words in M and if u is an elementary word, then the word $w_1 = xy$ is said to be a *reduct* of the word $w_0 = xuy$. We shall write $w_0 \Rightarrow w_1$ or $w_1 \Leftarrow w_0$ to indicate that w_1 is a reduct of w_0 . The relation $\Rightarrow$ will be called *reduction*, and $\Leftarrow$ will be called *reverse* reduction. The equivalence relation generated by $\Rightarrow$ will be denoted by $\cong$.

It is well-known, and easily verified, that $\cong$ is a congruence on the semigroup M , that is, $x_1 \cong x_2$ and $y_1 \cong y_2$ imply $x_1y_1 \cong x_2y_2$.

Also well-known is the fact that each equivalence class of $\cong$ contains precisely one element of F . We sketch a proof:

Firstly, there is at least one element of F in each equivalence class. For, if x_0 is a word in M then either x_0 is reduced or there is a reduct x_1 of x_0 . If x_1 is not reduced then it has a reduct x_2 , and so forth. Thus we have a chain of reductions

$$x_0 \Rightarrow x_1 \Rightarrow \ldots \Rightarrow x_n$$

which must end with a reduced word, because each x_i is shorter than its predecessor. By the definition of the equivalence relation $\cong$, we have $x_n \cong x_0$.

Conversely, suppose that x, y are words in F , and that we have $x \cong y$. By definition, there is a finite chain of reductions and reverse reductions from x to y :

$$x = z_0 \Leftarrow z_1 \ldots z_{n-1} \Rightarrow z_n = y \ .$$

The chain must begin with a reverse reduction and end with a reduction, since z_0, z_n are reduced. Now one shows that:

(1) If $z_{i-1} \Leftarrow z_i \Rightarrow z_{i+1}$ occurs in the chain, and if z_{i-1} or z_{i+1} is reduced, then we have $z_{i-1} = z_{i+1}$, and the chain may be shortened by deleting z_i and z_{i+1} .

(2) If $z_{i-1} \Leftarrow z_i \Rightarrow z_{i+1}$ occurs in the chain, and we have $z_{i-1} \neq z_{i+1}$, then there is a word z_i' which is a reduct of z_{i-1} and of z_{i+1} , so that we may replace $z_{i-1} \Leftarrow z_i \Rightarrow z_{i+1}$ in the chain by $z_{i-1} \Rightarrow z_i' \Leftarrow z_{i+1}$.

Finally, it is not difficult to show that by successive applications of these two rules the chain can be shortened to

$$x = z_0 = y \ .$$

Rule (2) above is often referred to as a "Diamond Lemma". For an extensive discussion of this kind of proof, we refer to M.H.A. Newman (1942).

The quotient semigroup $M/\cong$ is, of course, a group, namely the free group as it is usually defined. (See, for example, Magnus, Karrass and Solitar (1966).) The identity element is the equivalence class containing 1 , and for any x in M the inverse of the class containing x is the class containing $\bar{x}$. If X is the set defined in Section 1, then the classes containing elements of X freely generate $M/\cong$.

But we prefer to define the free group directly in terms of the elements of F , and the foregoing discussion makes it obvious how this is to be done.

DEFINITION. We define the operation of (group) *multiplication* on

F as follows: the *product* of two words x, y in F is the reduced word obtained from the concatenation xy in M by a sequence of zero or more reductions. The product will always be denoted by $x \cdot y$, the notation xy being reserved for the concatenation of x and y .

With this definition of multiplication, F becomes a group, obviously isomorphic to $M/\cong$. The identity element is 1 , and the inverse of a word x in F is the word $\bar{x}$. (If $\bar{x}$ has an elementary subword u , then $\bar{u}$ is also a subword of x , hence if x is in F so is $\bar{x}$.) Any set X defined as in Section 1 of this Chapter is a free generating set for the group F .

The present, very concrete, style of presentation for the free group goes back to papers of J. Nielsen (1921), and O. Schreier (1927).

If x, y are words in F , then clearly either the concatenation xy is in F (that is, we have $x \cdot y = xy$), or else the only elementary subword of the word xy in M comprises the last letter of x and the first letter of y . Thus the first reduct of xy must be a word x_1y_1 in M , where $x = x_1\bar{a}_1$ and $y = a_1y_1$ for some letter a_1 . Exactly the same reasoning may be applied to x_1y_1 , and so forth. Thus, the effect of n consecutive reductions of xy may be summarized by the removal from y of an initial subword $w = a_1a_2 \ldots a_n$ of length n , and the removal simultaneously of the final subword $\bar{w}$ from x .

DEFINITION. If $x = x_1\bar{w}$ and $y = wy_1$ are words in F , then the procedure of replacing the expression $x \cdot y$ by the expression $x_1 \cdot y_1$, or replacing the expression $x_1\bar{w} \cdot wy_1$ by the expression $x_1 \cdot y_1$, is called *cancellation*. (All of these expressions, of course, denote the same word in F .) The cancellation is called *trivial* if $w = 1$.

We may refer to the dot in such an expression as a "cancellation point": it indicates where cancellation is possible in an expression denoting a group product. Finally, we apply the term "reduced" to expressions (as distinct from words in M) as follows: a *reduced expression* is one in which no non-trivial cancellation is possible.

DEFINITION. In an arbitrary expression

$$x_1 \cdot x_2 \cdot \ldots \cdot x_n$$

denoting a group-product of words $x_1, x_2, \ldots, x_n$, we say that *cancellation dies* if $x_1 \neq 1$ and $x_n \neq 1$, and neither the first letter of x_1 nor the last letter of x_n can be removed by cancellation within the expression. If cancellation does not die within the expression, we say that *cancellation persists*.

The significance of an expression in which cancellation dies is that, if $x_1 \cdot x_2 \cdot \ldots \cdot x_n$ is such an expression, then the length of any expression $yx_1 \cdot x_2 \cdot \ldots \cdot x_n z$ is always the sum of the lengths of y, of $x_1 \cdot x_2 \cdot \ldots \cdot x_n$, and of z.

Let us note here a convention implicit in the notations and definitions introduced so far: the presence of cancellation-points in the above expressions implies that the words $x_1, \ldots, x_n, y, z$, and in particular also the words yx_1 and $x_n z$, are words in F.

We now extend this convention to all expressions (not just those containing cancellation-points) by restricting ourselves henceforth, except where the contrary is explicitly stated, to words in F. That is, *an expression without cancellation-points is reduced.* If we wish to use expressions (without cancellation-points) to denote words in M that may not be in F, we shall indicate this explicitly by referring to these expressions as being "in M", or writing, for example, "$xyz \in M$" rather than simply "xyz".

3. Coradicality

The following definition introduces a term which will be used constantly in our work.

DEFINITION. Two or more words $x_1, x_2, \ldots, x_n$ are said to be *coradical* if there are integers $m(1), m(2), \ldots, m(n)$, and a word z, such that

$$x_1 = z^{m(1)},\ x_2 = z^{m(2)},\ \ldots,\ x_n = z^{m(n)}\ .$$

The word z will be called a *common root* of the words $x_1, x_2, \ldots, x_n$. If all of the integers $m(1), \ldots, m(n)$ are non-negative, we shall say that $x_1, x_2, \ldots, x_n$ are *strictly coradical*.

Note that if two words x, y are not coradical, then neither of them can be 1. Again, if x, y are strictly coradical, then $xy = yx$ (by associativity). This statement has a converse:

THEOREM 2. *If x and y are words such that $xy = yx$, then x and y are strictly coradical.*

Proof. If $x = 1$ or $y = 1$ then trivially x and y are strictly coradical. If $\lambda(x) = \lambda(y)$ then by Corollary 1.1 we have

$x = y$, and again x, y are strictly coradical.

If $\lambda(xy) \leq 2$, then either $\lambda(x) = \lambda(y)$ or else one of x, y is the empty word 1 . We now argue by induction on $\lambda(xy) = \lambda(x) + \lambda(y)$.

Without loss of generality we may assume $\lambda(x) \geq \lambda(y)$. By Theorem 1 (with $u = y$, $v = x$) there is a word w such that $x = yw$ and $x = wy$. But this gives us $yw = wy$, and $\lambda(yw) = \lambda(x)$. Now unless $\lambda(y) = 1$ (in which case we know already that x, y are strictly coradical) we have $\lambda(x) < \lambda(x) + \lambda(y)$. The inductive hypothesis therefore applies to y and w , so there is a word z and there are non-negative integers m, n such that $y = z^m$ and $w = z^n$. But this implies $x = yw = z^{m+n}$, and $m+n \geq 0$, so x and y are strictly coradical, with z as a common root. //

THEOREM 3. *If* $x \neq 1$, $y \neq 1$, *and* x, y *are coradical, then precisely one of the following statements is true:*

(a) x, y *are strictly coradical,*

(b) $x, \bar{y}$ *are strictly coradical.*

Proof. The definitions assure us that at least one of the statements is true. Suppose that both are true. Without loss of generality, we may assume $\lambda(x) \geq \lambda(y)$. Clearly, then, y is a final subword of x (since *(a)* is true), and also $\bar{y}$ is a final subword of x (since *(b)* is true). That is, the word comprising the last $\lambda(y)$ letters of x (well-defined, since $\lambda(y) \leq \lambda(x)$) is both y and $\bar{y}$, hence $y = \bar{y}$, contradicting either the assumption $y \neq 1$ or the tacit assumption $y \in F$. //

THEOREM 4. *If* xy *appears in some expression (this implies* $xy \in F$ *), and if* x, y *are coradical (equivalently, if* $x, \bar{y}$ *are coradical), then* x, y *are strictly coradical.*

Proof. If x, y are coradical but not strictly coradical, then it follows from the definitions that $x \neq 1$, $y \neq 1$, and $x, \bar{y}$ are strictly coradical. Thus $x = z^m$, $y = \bar{z}^n$ for some $z \neq 1$ and integers $m, n \geq 1$. We have therefore $xy = z^m\bar{z}^n$, so xy has a subword $z\bar{z}$ with $z \neq 1$, contrary to the tacit assumption $xy \in F$. //

COROLLARY 4.1. *If* xy, y *are not coradical (equivalently, if* $xy, \bar{y}$ *are not coradical) then* x, y *are not coradical. Likewise, if* yx, y *are not coradical (or equivalently, if* $yx, \bar{y}$ *are not coradical) then* x, y *are not coradical.*

Proof. We need only prove the first statement. If we are given that xy, y are not coradical, then tacitly we are given also that xy is reduced. Therefore if x, y are coradical then by the above theorem they are strictly coradical. But then clearly xy and y are strictly coradical, which is a contradiction. //

The above fairly obvious corollary will be used very often indeed. Sometimes we use the following version, equivalent to Corollary 4.1 and obtained from it by iteration: if xy^n, y are not coradical (or if $y^n x$, $\bar{y}$ are not coradical), then x, y are not coradical.

COROLLARY 4.2. *If* yxz *and* yz *are not coradical, and if* zy *is reduced, then* x, zy *are not coradical.*

Proof. We prove the following assertion, which is a little stronger: if x, zy are coradical, and if yxz, yz are reduced, then yxz, yz are strictly coradical.

If $z = 1$, we have x, y coradical and yx reduced, therefore x, y are strictly coradical, by Theorem 4, and clearly yx, y (that is, yxz, yz) are also strictly coradical.

If $z \neq 1$, since yxz and zy are reduced we see that $yxzy$ is reduced, and therefore so is xzy. Therefore x, zy are strictly coradical, by Theorem 4, and we have

$$x = w^m \ , \quad zy = w^{n+1} \ ,$$

where m, n are non-negative integers and $w \neq 1$ is a common root. By Corollary 1.2, there are words u, v and non-negative integers r, s such that $w = uv$, $r + s = n$, $z = w^r u$, and $y = vw^s$. Thus we have

$$yxz = v(uv)^s (uv)^m (uv)^r u = (vu)^{m+n+1} \ ,$$

$$yz = v(uv)^s (uv)^r u = (vu)^{n+1}$$

so that yxz, yz are strictly coradical, with common root vu. //

THEOREM 5. *If* $x = yz$ *and* x, y *are coradical then* x, y, z *are strictly coradical. Likewise, if* $x = yz$ *and* x, z *are coradical then* x, y, z *are strictly coradical.*

Proof. Obviously the two statements are equivalent (invert and relabel), so we need only prove the first. Suppose x, y are coradical but not strictly coradical. Then it follows from the definitions that $x \neq 1$ and $y \neq 1$, and x, $\bar{y}$ are strictly coradical. Therefore $\bar{y}$, being of length at most $\lambda(x)$, is an initial subword of x. But so is y, therefore the word comprising the first $\lambda(y)$ letters of x is both y and $\bar{y}$, hence $y = \bar{y}$, and since $y \neq 1$ this contradicts $y \in F$. Therefore x, y are strictly coradical, and now obviously a common root for x, y is a common root for x, y, z. //

THEOREM 6. *If* $\bar{x}yx$ *appears in some expression (implying* $\bar{x}yx \in F$ *, by our conventions), then* x, y *are coradical only if* $x = 1$.

Proof. Suppose that x, y are coradical, and that $x \neq 1$. We observe that $\bar{x}y$ and yx both appear in $\bar{x}yx$, and now Theorem 4 implies that $\bar{x}, y$ are strictly coradical, and also that y, x are strictly coradical. But $y \neq 1$, otherwise $\bar{x}yx = \bar{x}x \notin F$, and now Theorem 3 implies either that $\bar{x}, y$ are not strictly coradical or that y, x are not strictly coradical. //

COROLLARY 6.1. *Let* n *be an arbitrary integer. If* $\bar{x}y^n x$ *appears in some expression, then* x, y *are coradical only if* $x = 1$.

Proof. If x, y are coradical then so are x, y^n. //

COROLLARY 6.2. *If* $\bar{x}yx\bar{y}$ *appears in some expression, then* x, y *are coradical only if* $x = 1$ *and* $y = 1$. //

COROLLARY 6.3. *Let* m, n *be positive integers. If* $\bar{x}y^m x^n \bar{y}$ *appears in some expression, then* x, y *are coradical only if* $x = 1$ *and* $y = 1$. //

Note that in Corollary 6.3 the restriction $m, n \geq 1$ is essential.

THEOREM 7. *If the words* $zx\bar{z}$, zy *are not coradical, and if* yz *is reduced, then* x *and* yz *are not coradical.*

Proof. This is trivially true if $z = 1$. Suppose $z \neq 1$. Then $x \neq 1$, by Theorem 6. Let a be the last letter of z, and write $z = z_1 a$. If x and yz are coradical, then a is the last letter of x or the last letter of $\bar{x}$, according to whether x and yz $(= yz_1 a)$ or $\bar{x}$ and yz are strictly coradical. If $x = x_1 a$ for some x_1, then we have $zx\bar{z} = z_1 a x_1 a \bar{a} \bar{z}_1$, which is not reduced, while if $x = \bar{a} x_1$ (so that $\bar{x} = \bar{x}_1 a$) then $zx\bar{z} = z_1 a \bar{a} x_1 \bar{a} \bar{z}_1$ which is again not reduced. //

THEOREM 8. *Suppose that* x_0, x_1 *are not coradical, and that cancellation persists in* $x_0 \cdot \bar{x}_1$ *and in* $x_1 \cdot x_0$. *Then there is a word* x_2 *in* F *such that* $x_0 = \bar{x}_1 x_2 x_1$, *and* x_1, x_2 *are not coradical.*

Proof. Firstly, we have $\lambda(x_1) < \lambda(x_0)$. For, if $\lambda(x_1) \geq \lambda(x_0)$

then, since cancellation persists in $x_1 \cdot \bar{x}_0$ and in $x_1 \cdot x_0$, we must have $x_1 = u\bar{x}_0 = vx_0$ for some u, v in F. Clearly $\lambda(u) = \lambda(v)$, so, by Theorem 1, we have $u = v$ and $\bar{x}_0 = x_0$. But this implies $x_0 = 1$, and then x_0, x_1 are coradical.

Now, $\lambda(x_1) < \lambda(x_0)$ and the persistence of cancellation in $x_0 \cdot \bar{x}_1$ and $x_1 \cdot x_0$ imply $x_0 = yx_1 = \bar{x}_1 z$ for some words y, z in F.

Next, we claim that $\lambda(x_1) < \lambda(y)$ $\bigl(= \lambda(z)\bigr)$. For, if $\lambda(x_1) \geq \lambda(y) = \lambda(z)$ then, by Theorem 1, there is a word w such that $\bar{x}_1 = yw$, that is, $x_1 = \bar{w}\bar{y}$, and $wz = x_1$. Therefore, again by Theorem 1, we have $\bar{w} = w$, so that $x_0 = yx_1 = \bar{x}_1 x_1$, hence x_0 is either not reduced or else equal to 1.

Finally, $\lambda(x_1) < \lambda(y) = \lambda(z)$ implies, by Theorem 1, the existence of a word x_2 such that $y = \bar{x}_1 x_2$ and $x_2 x_1 = z$. Therefore we have $x_0 = \bar{x}_1 x_2 x_1$, and by Theorem 6 the words x_1, x_2 are not coradical. //

THEOREM 9. *Suppose that the words x_0, x_1 are not coradical, and that cancellation persists in $x_0 \cdot \bar{x}_1$ and in $\bar{x}_1 \cdot x_0$. Then there are words u, v, not coradical, such that $x_0 = (uv)^m u$ and $x_1 = (uv)^n u$, where m, n are non-negative integers differing by 1.*

Proof. We argue by induction on $\max\bigl(\lambda(x_0), \lambda(x_1)\bigr)$. If $\lambda(x_0) = \lambda(x_1)$ then clearly $x_0 = x_1$, and x_0, x_1 are coradical. So $\lambda(x_0) \neq \lambda(x_1)$, and without loss of generality we may assume $\lambda(x_0) > \lambda(x_1)$.

The theorem holds "vacuously" for $\lambda(x_0) < 3$: clearly $\lambda(x_1) \geq 1$, and therefore $\lambda(x_0) \geq 2$, but $\lambda(x_0) = 2$ implies $x_0 = {x_1}^2$ so that x_0, x_1 are coradical.

If $\lambda(x_0) = 3$, then $\lambda(x_1) = 1$. For, otherwise we have $\lambda(x_1) = 2$, and then $x_0 = a_1 a_2 a_3$ (where a_1, a_2, a_3 are letters) implies $x_1 = a_1 a_2 = a_2 a_3$ and hence $a_1 = a_2$ and $a_2 = a_3$, so that $x_0 = {a_1}^3$ and $x_1 = {a_1}^2$. But $\lambda(x_0) = 3$, $\lambda(x_1) = 1$ imply $x_0 = x_1 v x_1$, where $\lambda(v) = 1$ and obviously $v \neq x_1$. Thus our assertion holds with $u = x_1$, $m = 1$, $n = 0$.

Consider now the general case. We have $x_0 = yx_1 = x_1 z$ for some words y, z. If $\lambda(x_0) \geq 2\lambda(x_1)$, so that $\lambda(x_1) \leq \lambda(y) = \lambda(z)$, then Theorem 1 gives $y = x_1 v$, $z = vx_1$ for some word v, and now $x_0 = yx_1 = x_1 v x_1$. Here x_1, v are not coradical, else they are strictly coradical by Theorem 4, and then so are x_0, x_1. Therefore our assertion holds with $u = x_1$, $m = 1$, $n = 0$. $\{$Clearly $\lambda(x_0) \neq 2\lambda(x_1)\}$. On the other hand, if $\lambda(x_0) < 2\lambda(x_1)$, so that $\lambda(x_1) > \lambda(y) = \lambda(z)$, then Theorem 1 gives $x_1 = yx_2 = x_2z$ for some word x_2, and $x_0 = yx_2z$. Now x_1, x_2 are not coradical, else Theorem 5 shows that x_1, y, x_2 are strictly coradical, and since $yx_2 = x_2z$ any common root of x_1, y, x_2 serves also for z, so that $x_0 = yx_2z$ and $x_1 = yx_2$ are strictly coradical, contrary to hypothesis. Therefore $y \neq 1$ and $z \neq 1$, so that $\lambda(x_2) < \lambda(x_1) < \lambda(x_0)$, and now x_1 and x_2 satisfy the inductive hypothesis: cancellation persists in $x_1 \cdot \bar{x}_2$ and in $\bar{x}_2 \cdot x_1$, x_1 and x_2 are not coradical, and

$$\lambda(x_1) = \max\{\lambda(x_1), \lambda(x_2)\} < \lambda(x_0) = \max\{\lambda(x_0), \lambda(x_1)\} .$$

Therefore there are words u, v, not coradical, and non-negative integers m, n differing by 1, such that $x_1 = (uv)^m u$ and $x_2 = (uv)^n u$. Obviously $m = n + 1$, since $\lambda(x_1) > \lambda(x_2)$. But now $x_1 = x_2z$ gives $(uv)^{n+1}u = (uv)\ uz$, hence $vu = z$. Therefore we have

$$x_0 = x_1z = (uv)^{n+1}u(vu) = (uv)^{n+2}u ,$$

and our assertion holds for x_0 and x_1. //

COROLLARY 9.1. *If* $\max\{\lambda(x_0), \lambda(x_1)\} \leq 2\min\{\lambda(x_0), \lambda(x_1)\}$ *in the above theorem, then* m, n *are both positive.* //

THEOREM 10. *Suppose that* xy, yz *are coradical. Then either* x, y, z *are coradical, or else there are words* u, v*, not coradical, such that* $x = \{(uv)^{k+1}u\}^m uv$, $y = (uv)^k u$, $z = vu\{(uv)^{k+1}u\}^n$, *where* k, m, n *are non-negative integers.*

Proof. Assume that x, y, z are not coradical. Then clearly $y \neq 1$, and it follows that xy, yz are strictly coradical: otherwise xy and $\bar{z}\bar{y}$ are strictly coradical, so that the shorter of them is a final subword of the longer, and and this in turn implies that they have in common their final subword of length $\lambda(y)$, that is, $y = \bar{y}$, and this contradicts either $y \neq 1$ or the tacit assumption $y \in F$. So we have $xy = w^{m+1}$, $yz = w^{n+1}$ for some word $w \neq 1$ and non-negative integers m, n. By Corollary 1.2, there are words u_1, v_1 and non-negative integers r, s such that $w = u_1v_1$, $r + s = m$, $x = w^r u_1$, and $y = v_1w^s$. Now $w^{n+1} = yz$ gives $(u_1v_1)^{n+1} = v_1(u_1v_1)^s z$, and this implies $s = 0$: otherwise we may equate initial subwords of length $\lambda(u_1) + \lambda(v_1)$ to give $u_1v_1 = v_1u_1$, so that u_1, v_1 are strictly coradical, by Theorem 2, and in turn so are $x = (u_1v_1)^r u_1$,

$y = v_1(u_1v_1)^s$ and $z = (u_1v_1)^{n-s}u_1$ (this last equation comes from $w^{n+1} = yz$ by substituting $w = v_1u_1$ and cancelling). But $s = 0$ implies $m = r$, $y = v_1w^s = v_1$ and $w = u_1y$, and now $w^{n+1} = yz$ implies (by Theorem 1, with $\lambda(y) \le \lambda(w)$) that $w = yu_2$ for some u_2. Now w, y are not coradical, else u_1 and v_1 $(= y)$ are strictly coradical by Theorem 5, and we have seen that this is impossible. Therefore w and y satisfy the hypotheses of Theorem 9, and there are words u, v, not coradical, such that $w = (uv)^{k+1}u$ and $y = (uv)^ku$. Now $w = u_1y$ implies $u_1 = uv$, and we have $x = w^mu_1 = \big((uv)^{k+1}u\big)^m uv$, while $w^{n+1} = yz$ implies $\big((uv)^{k+1}u\big)^{n+1} = (uv)^kuz$ and by cancellation $z = vu\big((uv)^{k+1}u\big)^n$. //

COROLLARY 10.1. *If* $x^m = y^n$, *where* m, n *are non-zero integers and* x, y *are words in* F, *then* x, y *are coradical.*

Proof. Clearly, it is sufficient to prove our assertion for m, n positive.

Equating lengths, we have $m\lambda(x) = n\lambda(y)$. If $m = n$, then $\lambda(x) = \lambda(y)$, and $x = y$ by Theorem 1, so that x, y are coradical.

If $m \ne n$, without loss of generality we may assume $m > n$, and hence $\lambda(x) < \lambda(y)$. Now x is both an initial and a final subword of y^n, and hence, by Theorem 1, of y. That is, there are words u_1, v_1 such that $y = u_1x = xv_1$. By the above theorem, either x, u_1, v_1 are coradical or else there are words u, v, not coradical, and a non-negative integer k such that

$$u_1 = \big((uv)^{k+1}u\big)uv \quad \text{and} \quad x = (uv)^ku .$$

The first alternative implies that x, u_1, v_1 are strictly coradical by Theorem 4, and hence that x, y are coradical.

The second alternative gives $y = u_1x = \big((uv)^{k+1}u\big)^{s+1}$, hence

$$\big((uv)^ku\big)^m = \big((uv)^{k+1}u\big)^{n(s+1)}$$

and by cancellation

$$\big((uv)^ku\big)^{m-1} = vu\big((uv)^{k+1}u\big)^{n(s+1)-1} .$$

Clearly $m \ne 1$, since $vu \ne 1$. If $k \ne 0$ we have (comparing initial subwords) $uv = vu$ so that u, v are coradical, contrary to hypothesis. Therefore $k = 0$, and we have

$$u^m = (uvu)^{n(s+1)} .$$

Equating lengths, we have $m\lambda(u) = n(s+1)\lambda(uvu)$ and hence

$$\left(\frac{m}{n(s+1)} - 2\right)\lambda(u) = \lambda(v) .$$

Therefore $n(s+1)$ divides m , and *a fortiori* n divides m . But this implies $x^{m/n} = y$, so that x, y are again coradical. //

CHAPTER 2

1. Excerpts in one variable

Our task in this Chapter will be to collect examples of expressions in which cancellation-points appear, but in which (under certain conditions) cancellation dies. Anticipating the role of these expressions in our main proof, we may call them "excerpts". We shall be interested especially in finding lower bounds for the lengths of these "excerpts", stated in terms of the lengths of the variables appearing in them.

The simplest of these expressions, those in which only one variable appears, are given by the following elementary and familiar theorem.

THEOREM 11. *If* $x \neq 1$ *, then there are uniquely defined words* y, z *such that* $x = zy\bar{z}$ *,* $y \neq 1$ *, and* $y \cdot y = y^2$ *.*

Proof. If $x \neq 1$, we may write $x = a_1 a_2 \ldots a_n$, where $n \geq 1$ and the a_i are letters (not necessarily all distinct). There are integers t such that $a_t \neq \bar{a}_{n-t+1}$: otherwise $n = 2m$ $(m \geq 1)$ implies $a_{m+1} = \bar{a}_{2m-m-1+1} = \bar{a}_m$ and x has an elementary subword $a_m a_{m+1} = a_m \bar{a}_m$, while $n = 2m - 1$ $(m \geq 1)$ implies $a_m = \bar{a}_{2m-1-m+1} = \bar{a}_m$, so that the mapping $a \longmapsto \bar{a}$ of A , contrary to its definition, has a fixed point. Let k be the smallest of the integers t as described. Then $n - k + 1$ is the largest, and now we have $x = zy\bar{z}$, where $z = a_1 a_2 \ldots a_{k-1}$, $y = a_k \ldots a_{n-k+1}$, and by construction $y \neq 1$ and $y \cdot y = yy = y^2$. The uniqueness of y and z follows from Theorem 1. //

COROLLARY 11.1. *If* $x \neq 1$ *then cancellation dies in the expressions* $x \cdot x$ *,* $x \cdot x \cdot x$ *, etc., and we have* $\lambda(x \cdot x) \geq \lambda(x) + 1$ *,* $\lambda(x \cdot x \cdot x) \geq \lambda(x) + 2$ *, etc.*

Proof. With y, z as defined in the above proof, we have $x \cdot x = zy^2\bar{z}$, $x \cdot x \cdot x = zy^3\bar{z}$, etc., hence $\lambda(x \cdot x) = \lambda(x) + \lambda(y) \geq \lambda(x) + 1$, etc. //

2. Circular words and expressions

DEFINITION. A word x is *cyclically reduced* if the last letter of x (if any) is not inverse to the first letter.

Equivalently, x is cyclically reduced if $x \cdot x = x^2$. Note that if $x \neq 1$ is not cyclically reduced and $y \neq 1$ is a word such that $\bar{y}x$ is reduced, then $x \cdot y$ is likewise reduced, that is, $x \cdot y = xy$.

With the usual definition of conjugacy, that is, $x^y = \bar{y} \cdot x \cdot y$, the above theorem shows that every word x has a cyclically reduced conjugate. But unless $x = a^n$ for some letter a and integer n (this includes $x = 1$), the cyclically reduced conjugate is not unique. For most purposes, this non-uniqueness is irrelevant: any of the cyclically reduced conjugates will do - and often "all of them at once" will do even better, as one wishes to shift the argument from one cyclically reduced conjugate to another. This motivates our definition (inspired by Tartakovskii) of "circular words".

DEFINITION. If x is a cyclically reduced word in F , then the *circular word derived from* x , which we denote by x^* , comprises the same letters as x , but is written along the circumference of a circle with orientation (for example, anticlockwise) rather than linearly from left to right.

Of course, only the topological properties of a circle are required here: the "cyclic" ordering of the letters following the orientation of the circle is important, but such things as the length and curvature of the "circle"'s perimeter are not. Thus the letters are to be thought of as arranged like beads on an (oriented) elastic necklace - or, more abstractly, they are used as labels of consecutive sub-arcs of a homeomorphic image of an oriented circle. Starting at an arbitrary letter of x^* and traversing the "circle" once in the sense of its orientation, we read off the successive letters of one of the cyclically reduced conjugates of x . Depending on the starting point, every cyclically reduced conjugate may be read off in this way.

If x is not cyclically reduced, we define x^* to be the circular word derived from any cyclically reduced conjugate of x : by Theorem 11, x^* is thus defined for every x in F . Extending the definition further, we shall use the same "asterisk" notation for expressions, reduced or otherwise, to denote the circular word derived from the word in F denoted by the expression minus the asterisk.

The *cyclically reduced length* of an arbitrary word x in F is now defined in the obvious way, as the length of x^* (that is, the number of letters in the circular word x^*). It will be denoted by $\lambda(x^*)$. Obviously $\lambda(x) \geq \lambda(x^*)$ for each x in F , and by Theorem 11 we have $\lambda(x^*) = 0$ if and only if $\lambda(x) = 0$.

Evidently, the circular word x^* so defined answers admirably well our wish for a concrete, "combinatorial" representation for the conjugacy class represented by x. The "composition" operator of Tartakovskii may be represented in the style of Lyndon and Schupp as the identification, or "sticking together", of two such "circles" at a common vertex point (between consecutive letters), the product being read off along the outside of the resulting diagram. (To make this unambiguous, the whole process must be carried out in the plane, with a standardized orientation of the circles, for example, all anticlockwise when viewed from "above" the plane.)

3. Excerpts in two or more variables

We proceed with our study of "excerpts" by considering those in two or more variables. As one might expect, in comparison with the one-variable case the variables which occur must usually satisfy rather more stringent conditions to ensure that cancellation dies in these expressions. It turns out that these restrictions can be formulated quite simply, in terms of coradicality.

We shall, on occasion, refer to the length of an expression. By this we shall mean, of course, the length of the word in F denoted by that expression.

THEOREM 12. *Cancellation dies in the expression* $\bar{y}x\cdot y\bar{x}$ *if and only if* x *and* y *are not coradical, and in this case we have* $\lambda(\bar{y}x\cdot y\bar{x}) \geq 4$.

Proof. Suppose x, y are coradical. Then $\bar{x}, y$ are strictly coradical, by Theorem 4. Therefore $y\bar{x} = \bar{x}y$, so that in the above expression the whole word $\bar{y}x$ on the left may be cancelled against the whole word $y\bar{x}$ on the right. That is, cancellation persists in $\bar{y}x\cdot y\bar{x}$.

Suppose now that x, y are not coradical. Then $x \neq 1$ and $y \neq 1$, so that each of the words $\bar{y}x$, $y\bar{x}$ has at least two letters. We shall prove that no cancellation can alter this state of affairs, thereby establishing both the non-persistence of cancellation in $\bar{y}x\cdot y\bar{x}$ and the lower bound on $\lambda(\bar{y}x\cdot y\bar{x})$.

If cancellation dies in $x\cdot y$, the desired conclusion follows directly: there are words $x_1 \neq 1$, $y_1 \neq 1$ and z such that $x = x_1z$, $y = zy_1$, and $x_1\cdot y_1 = x_1y_1$, so that

$$\bar{y}x\cdot y\bar{x} = \bar{y}_1zx_1y_1\bar{z}\bar{x}_1$$

and clearly $\lambda(\bar{y}x \cdot y\bar{x}) = 2\big(\lambda(x_1)+\lambda(y_1)+\lambda(z)\big) \geq 4$, since $\lambda(x_1) \geq 1$ and $\lambda(y_1) \geq 1$.

If cancellation persists in $x \cdot y$, then either $\bar{x}$ is an initial subword of y or else $\bar{y}$ is a final subword of x . (Not both, else x, y are coradical.) But if $y = \bar{x}y_1$ then $\bar{y}x \cdot y\bar{x} = \bar{y}_1 x \cdot y_1 \bar{x}$, where x, y_1 are not coradical, by Corollary 4.1, while if $x = x_1\bar{y}$ then $\bar{y}x \cdot y\bar{x} = \bar{y}x_1 \cdot y\bar{x}_1$, where x_1, y are not coradical, again by Corollary 4.1. That is, after we cancel x on the left or y on the right, what remains is another expression of the same form as the original one and satisfying the same conditions.

Since x, y (and the corresponding x_1, y_1 , etc.) are of positive length, and $\lambda(\bar{y}x) = \lambda(y\bar{x})$ is finite, it is clear that every cancellation in the original expression is given by a (finite) sequence of cancellations such as those we have considered. Our theorem is therefore proved. //

If $x = \bar{a}_2\bar{a}_1{}^n$ and $y = \bar{x}^m a_1$, where m, n are positive integers and a_1, a_2 are letters, then

$$\bar{y}x \cdot y\bar{x} = \bar{a}_1 x \cdot a_1 \bar{x} = \bar{a}_1\bar{a}_2 a_1 a_2 \ ,$$

so the lower bound on $\lambda(\bar{y}x \cdot y\bar{x})$ can be attained for arbitrarily large values of $\min\big(\lambda(x), \lambda(y)\big)$. The next theorem is a strict generalization of the last one.

THEOREM 13. *If m, n are positive integers, cancellation dies in the expression $\bar{y}x^m \cdot y^n \bar{x}$ if and only if x, y are not coradical, and in this case we have*

$$\lambda(\bar{y}x^m \cdot y^n \bar{x}) \geq (m-1)\lambda(x) + (n-1)\lambda(y) + 4 \ .$$

Proof. Suppose x, y are coradical. Then $x, \bar{y}$ are strictly coradical, by Theorem 4. Therefore $\bar{y}x^n$ and $y^n\bar{x}$ are powers, with opposite sign, of some common root, and clearly cancellation persists in the above expression.

Suppose now that x, y are not coradical.

If cancellation dies in $x \cdot y$, then there are words $x_1 \neq 1$, $y_1 \neq 1$ and z such that $x = x_1 z$, $y = \bar{z}y_1$, and $x_1 \cdot y_1 = x_1 y_1$. Substituting for x and y , we have therefore

$$\begin{aligned}\bar{y}x^m \cdot y^n \bar{x} &= \bar{y}_1 z(x_1 z)^m \cdot (\bar{z}y_1)^n \bar{z}\bar{x}_1 \\ &= \bar{y}_1(zx_1)^m(y_1\bar{z})^n \bar{x}_1 \ ,\end{aligned}$$

and the length of this is

$$m\lambda(x) + n\lambda(y) + \lambda(x_1) + \lambda(y_1) ,$$

which is at least $(m-1)\lambda(x) + (n-1)\lambda(y) + 4$, since $\lambda(x) \geq \lambda(x_1) \geq 1$ and $\lambda(y) \geq \lambda(y_1) \geq 1$.

On the other hand, if cancellation persists in $x \cdot y$, then either $\bar{x}$ is an initial subword of y , or else $\bar{y}$ is a final subword of x . (Not both, else x, y are coradical.) Without loss of generality, we may assume that $\bar{y}$ is a final subword of x , since the other case may be dealt with by the same argument (interchange m and n , y and $\bar{x}$, and invert the resulting expression). Accordingly we write $x = x_1\bar{y}^s$, choosing $s \geq 1$ as large as possible. Here s is well-defined, since $y \neq 1$. With this substitution, we have

$$\begin{aligned} \bar{y}x^m \cdot y^n \bar{x} &= \bar{y}(x_1\bar{y}^s)^m \cdot y^n y^s \bar{x}_1 \\ &= (\bar{y}x_1\bar{y}^{s-1})^{m-1}\bar{y}x_1 \cdot y^n \bar{x}_1 , \end{aligned}$$

where x_1, y are not coradical, by Corollary 4.1. If cancellation dies in $x_1 \cdot y$ then the length of this expression is

$$\begin{aligned} &(m-1)\lambda(\bar{y}x_1\bar{y}^{s-1}) + n\lambda(y) + \lambda(x_1) + \lambda(x_1 \cdot y) \\ &\qquad = (m-1)\lambda(x) + (n-1)\lambda(y) + \lambda(x_1) + \lambda(y) + \lambda(x_1 \cdot y) \\ &\qquad \geq (m-1)\lambda(x) + (n-1)\lambda(y) + 4 , \end{aligned}$$

since $\lambda(x_1 \cdot y) \geq 2$ and $\lambda(x_1), \lambda(y) \geq 1$. (Strictly speaking, to prove this we should introduce words $x_2 \neq 1$, $y_1 \neq 1$ and z such that $x_1 = x_2 z$, $y = \bar{z}y_1$ and $x_2 \cdot y_1 = x_2 y_1$, and so on.) If cancellation persists in $x_1 \cdot y$ then $\bar{x}_1$ is an initial subword of y ; the other alternative, namely that $\bar{y}$ is a final subword of x_1 , is ruled out because s was chosen as large as possible. But now we have $y = \bar{x}_1 y_1$ for some word y_1 , and x_1, y_1 are not coradical, by Corollary 4.1 (we know already that x_1, y are not coradical). Making this substitution for y in $\bar{y}x_1 \cdot y^n \bar{x}_1$, we find

$$\begin{aligned} \bar{y}x^m \cdot y^n \bar{x} &= (\bar{y}x_1\bar{y}^{s-1})^{m-1}\bar{y}x_1 \cdot y^n \bar{x}_1 \\ &= (\bar{y}x_1\bar{y}^{s-1})^{m-1}\bar{y}_1 x_1 \cdot y_1\bar{x}_1(y_1\bar{x}_1)^{n-1} \end{aligned}$$

and by Theorem 12 cancellation dies in $\bar{y}_1 x_1 \cdot y_1 \bar{x}_1$ and $\lambda(\bar{y}_1 x_1 \cdot y_1 \bar{x}_1)$ is at least 4 . *A fortiori*, cancellation dies in $\bar{y}x^m \cdot y^n x$, and the length of this expression is

$$\begin{aligned} &(m-1)\lambda(\bar{y}x_1\bar{y}^{s-1}) + (n-1)\lambda(y_1\bar{x}_1) + \lambda(\bar{y}_1 x_1 \cdot y_1 \bar{x}_1) \\ &\qquad = (m-1)\lambda(x) + (n-1)\lambda(y) + \lambda(\bar{y}_1 x_1 \cdot y_1 \bar{x}_1) \\ &\qquad \geq (m-1)\lambda(x) + (n-1)\lambda(y) + 4 . \qquad // \end{aligned}$$

In the above proofs of Theorem 12 and Theorem 13, we discern a motif which will recur throughout the present work: in studying the effects of cancellation in an expression $\text{---}\, x\cdot y\, \text{---}$, we "transcribe" one or both of the variables adjoining the cancellation-point, for example by writing $x = x_1\bar{y}$ or $y = \bar{x}y_1$, and then study the expression that results after the indicated substitution and cancellation have been carried out. In the present illustration, if x, y are known to be not coradical, then x_1, y (respectively x, y_1) are also not coradical, by Corollary 4.1.

To make this quite formal, we define "transcription" and "transcribe" as technical terms.

DEFINITION. If we are given an expression $\text{---}\, x\cdot y\, \text{---}$, where the dashes denote unspecified parts of the expression, then we perform the *transcription* denoted by $x \to x_1\bar{y}$, or, more briefly, we *transcribe* $x \to x_1\bar{y}$, as follows:

(a) replace x wherever it occurs in the given expression by $x_1\bar{y}$ (where $x = x_1\bar{y}$ is tacitly assumed), and then

(b) cancel $\bar{y}$ against y in every context $x_1\bar{y}\cdot y$ that appears in the expression resulting after step (a).

A transcription $y \to \bar{x}y_1$ is defined in the same way: replace y throughout by $\bar{x}y_1$, and cancel x against $\bar{x}$ wherever possible in the resulting expression. (Here $y = \bar{x}y_1$ is tacitly assumed.)

We define also the *simultaneous* transcription, denoted for example by $(x, y) \to (x_1z, \bar{z}y_1)$, as the procedure of replacing x by x_1z and y by $\bar{z}y_1$ throughout the given expression $\text{---}\, x\cdot y\, \text{---}$, and then cancelling z against $\bar{z}$ in every context $x_1z\cdot\bar{z}y_1$ that results. (Again, $x = x_1z$ and $y = \bar{z}y_1$ are tacitly assumed.) The composition of two transcriptions is again called a transcription.

On occasion, we shall also use transcriptions such as $x \to x\bar{y}$ or $y \to \bar{x}y$. A transcription $x \to x\bar{y}$ is to be interpreted as a transcription $x \to x_1\bar{y}$, where x_1 is a variable not previously used, followed by a "re-labelling" from x_1 to x (this is consistent, since after the transcription $x \to x_1\bar{y}$ there are no occurrences of the "old" x in the expression being studied).

THEOREM 14. *Let m be a positive integer. Then cancellation dies in the expression $\bar{y}x^m\cdot yx$ if and only if $x \neq 1$. If $x \neq 1$ and $y \neq 1$ then we have both $\lambda(\bar{y}x^m\cdot yx) \geq m\lambda(x) + 2$ and $\lambda(\bar{y}x^m\cdot yx) \geq (m-1)\lambda(x) + 4$.*

Proof. If $x = 1$ then obviously cancellation persists. If

$x \neq 1$ and $y = 1$ then the expression becomes $x^m \cdot x$, in which cancellation dies by Corollary 11.1, though the length may be less than the bound given here.

Suppose, then, that $x \neq 1$ and $y \neq 1$. It is clear that x, y cannot be coradical, else by Theorem 4 we have $x, \bar{y}$ strictly coradical and also x, y strictly coradical (since $\bar{y}x$ and xy are both reduced), and this contradicts Theorem 3.

We may transcribe $y \to \bar{x}^s y$, where the integer s is chosen as large as possible. The integer s is well-defined, since $\lambda(x) \geq 1$ and $\lambda(y)$ is finite; if $\bar{x}$ is not an initial subword of y then obviously we have $s = 0$. This gives

$$\bar{y}x^m \cdot yx = \bar{y}_1 x^m \cdot y_1 x$$

where x, y_1 are not coradical if $s \neq 0$, and in any case $x \neq 1$ and $y_1 \neq 1$.

If cancellation dies in $x \cdot y_1$ then we have

$$\begin{aligned} \lambda(\bar{y}x^m \cdot yx) &= \lambda(\bar{y}_1 x^{m-1}) + \lambda(x \cdot y_1) + \lambda(x) \\ &= m\lambda(x) + \lambda(y_1) + \lambda(x \cdot y_1) \\ &\geq m\lambda(x) + 3 \\ &\geq (m-1)\lambda(x) + 4 , \end{aligned}$$

since $\lambda(x), \lambda(y_1) \geq 1$ and $\lambda(x \cdot y_1) \geq 2$.

If cancellation persists in $x \cdot y_1$ then we may transcribe $x \to x_1 \bar{y}_1$. The other alternative, namely that we may transcribe $y_1 \to \bar{x} y_2$ for some y_2, is ruled out because this would imply that s was not largest possible, contrary to its definition. So we have

$$\bar{y}x^m \cdot yx = (\bar{y}_1 x_1)^{m-1} \bar{y}_1 x_1 \cdot x_1 \bar{y}_1$$

where x_1, y_1 are not coradical, by Corollary 4.1, and in particular $x_1 \neq 1$, so that by Corollary 11.1 cancellation dies in $x_1 \cdot x_1$. Therefore we have

$$\begin{aligned} \lambda(\bar{y}x^m \cdot yx) &\geq (m-1)\lambda(\bar{y}_1 x_1) + 2\lambda(y_1) + \lambda(x_1) + 1 \\ &= m\lambda(x) + \lambda(y_1) + 1 \\ &\geq m\lambda(x) + 2 , \quad \text{since } \lambda(y) \geq 1 , \\ &\geq (m-1)\lambda(x) + 4 , \end{aligned}$$

since $\lambda(x) = \lambda(x_1) + \lambda(y_1) \geq 2$. //

If a, b are letters (not necessarily distinct) and z is a word such that $\bar{z}a\bar{z}\bar{b}$ and azb are reduced, then for $x = \bar{z}az\bar{b}$ and $y = (b\bar{z}\bar{a}z)^s b$ we have $\bar{y}x^m \cdot yx = (\bar{b}\bar{z}az)^{m-1}\bar{b}\bar{z}a^2 z\bar{b}$, a word whose length is precisely $m\lambda(x) + 2$. Since $\lambda(z)$ and s can be made arbitrarily large, this bound is sharp irrespective of any restriction on $\min(\lambda(x), \lambda(y))$. Examples for which $\lambda(\bar{y}x^m \cdot yx) = (m-1)\lambda(x) + 4$ are easily found; clearly they must all satisfy $\lambda(x) \leq 2$.

THEOREM 15. *Let* m, n *be positive integers. Then cancellation dies in the expression* $\bar{y}x^m \cdot y^n x$ *if and only if* $x \neq 1$. *If* $x \neq 1$ *and* $y \neq 1$ *then we have* $\lambda(\bar{y}x^m \cdot y^n x) \geq (m-1)\lambda(x) + (n-1)\lambda(y) + 4$.

Proof. If $n = 1$, this result is given by Theorem 14, so we may assume $n > 1$ and proceed by induction on n .

As in Theorem 14, cancellation obviously persists if $x = 1$, and if $x \neq 1$, $y = 1$ we have $x^m \cdot x$, in which cancellation dies by Corollary 11.1.

Suppose, then, that $x \neq 1$ and $y \neq 1$. Then x, y are not coradical, else by Theorem 4 we have $x, \bar{y}$ strictly coradical and x, y strictly coradical (since $\bar{y}x$ and yx are reduced), contradicting Theorem 3.

If cancellation dies in $x \cdot y$, we have

$$\begin{aligned} \lambda(\bar{y}x^m \cdot y^n x) &= \lambda(\bar{y}x^{m-1}) + \lambda(x \cdot y) + \lambda(y^{n-1}x) \\ &= m\lambda(x) + n\lambda(y) + \lambda(x \cdot y) \\ &\geq (m-1)\lambda(x) + (n-1)\lambda(y) + 4 \end{aligned}$$

since $\lambda(x) \geq 1$, $\lambda(y) \geq 1$, and $\lambda(x \cdot y) \geq 2$.

If cancellation persists in $x \cdot y$, we consider separately the cases $\lambda(x) > \lambda(y)$ and $\lambda(x) < \lambda(y)$. (The case $\lambda(x) = \lambda(y)$ does not arise, since x, y are not coradical.)

In the case $\lambda(x) > \lambda(y)$, we transcribe $x \to x_1\bar{y}$. This gives

$$\bar{y}x^m \cdot y^n x = (\bar{y}x_1)^{m-1}\bar{y}x_1 \cdot y^{n-1}x_1\bar{y}$$

where x_1, y are not coradical, by Corollary 4.1, and in particular we have $x_1 \neq 1$ and $y \neq 1$. Since n was assumed greater than 1 , by the inductive hypothesis cancellation dies in $\bar{y}x_1 \cdot y^{n-1}x_1$, and $\lambda(\bar{y}x_1 \cdot y^{n-1}x_1) \geq (n-2)\lambda(y) + 4$, so we have

$$\begin{aligned} \lambda(\bar{y}x^m \cdot y^n x) &= (m-1)\lambda(\bar{y}x_1) + \lambda(\bar{y}x_1 \cdot y^{n-1}x_1) + \lambda(y) \\ &\geq (m-1)\lambda(x) + (n-1)\lambda(y) + 4 . \end{aligned}$$

In the case $\lambda(x) < \lambda(y)$, we transcribe $y \to \bar{x}y_1$, to obtain

$$\bar{y}x^m \cdot y^n x = \bar{y}_1 x^m \cdot y_1 \bar{x} y_1 (\bar{x}y_1)^{n-2}x$$

where x, y_1 are not coradical. (The expression on the right makes sense, since $n > 1$.) By Theorem 13, cancellation dies in $\bar{y}_1 x^m \cdot y_1 \bar{x}$, and $\lambda(\bar{y}_1 x^m \cdot y_1 \bar{x}) \geq (m-1)\lambda(x) + 4$, so we have

$$\lambda(\bar{y}x^m \cdot y^n x) \geq (m-1)\lambda(x) + 4 + \lambda(y_1) + (n-2)\lambda(\bar{x}y_1) + \lambda(x)$$
$$= (m-1)\lambda(x) + (n-1)\lambda(y) + 4 . \qquad //$$

If $n > 1$, we may put $x = \bar{a}\bar{b}^t$, $y = (b^t a)^s b$, where a, b are letters and s, t are arbitrary positive integers, and verify that the above bound is attained for arbitrarily large values of $\min\{\lambda(x), \lambda(y)\}$.

THEOREM 16. *Let m be a positive integer. Cancellation dies in $yx^m \cdot yx$ if and only if $x \neq 1$ or $y \neq 1$, and then we have* $\lambda(yx^m \cdot yx) \geq \lambda(x) + \lambda(y) + 1$.

Proof. Obviously cancellation persists if $x = 1$ and $y = 1$. If $x \neq 1 = y$ or $x = 1 \neq y$, the fact that cancellation dies, and the bound on $\lambda(yx^m \cdot yx)$, are given by Corollary 11.1.

Suppose, then, that $x \neq 1$ and $y \neq 1$.

If $m = 1$, the assertions of the theorem are again given by Corollary 11.1: cancellation dies in $yx \cdot yx$, and $\lambda(yx \cdot yx) \geq \lambda(yx) + 1 = \lambda(x) + \lambda(y) + 1$. We now assume $m > 1$, and proceed by induction on m .

If cancellation dies in $x \cdot y$, *a fortiori* it dies in $yx^m \cdot yx$, and we have

$$\lambda(yx^m \cdot yx) = m\lambda(x) + \lambda(y) + \lambda(x \cdot y)$$
$$> \lambda(x) + \lambda(y) + 1 .$$

If cancellation persists in $x \cdot y$ then $\lambda(x) \neq \lambda(y)$, else $x = \bar{y} \neq 1$, so that yx cannot be in F . We consider separately the cases $\lambda(x) > \lambda(y)$ and $\lambda(x) < \lambda(y)$.

In the case $\lambda(x) > \lambda(y)$, we transcribe $x \to x_1\bar{y}$. This gives

$$yx^m \cdot yx = y(x_1\bar{y})^{m-1}x_1 \cdot x_1\bar{y} ,$$

where $x_1 \neq 1$. (In fact x_1, y are not coradical, since the reduced expression yx becomes $yx_1\bar{y}$, and $y \neq 1$.) Now by Corollary 11.1 cancellation dies in $x_1 \cdot x_1$, and we have

$$\lambda(yx^m \cdot yx) \geq m\lambda(x_1) + (m+1)\lambda(y) + 1$$
$$= m\lambda(x) + \lambda(y) + 1 .$$

In the case $\lambda(x) < \lambda(y)$, we transcribe $y \to \bar{x}y_1$, where $y_1 \neq 1$, and we have

$$yx^m \cdot yx = \bar{x}y_1x^{m-1} \cdot y_1x$$

and now by the inductive hypothesis we know that cancellation dies, and

$$\begin{aligned}\lambda(yx^m \cdot yx) &\geq \lambda(x) + \lambda(x) + \lambda(y_1) + 1 \\ &= \lambda(x) + \lambda(y) + 1 \ . \qquad //\end{aligned}$$

Perhaps surprisingly, the bound is sharp: if $x = a^r$ and $y = \bar{a}^{mr}b$, where a, b are letters and r is an arbitrary positive integer, then $\min\{\lambda(x), \lambda(y)\} = r$, and $yx^m \cdot yx = \bar{a}^{mr}b^2a^r$, which has length precisely $\lambda(x) + \lambda(y) + 1$, irrespective of m or r .

THEOREM 17. *Let m, n be integers greater than 1 . Cancellation dies in the expression $yx^m \cdot y^n x$ if and only if $x \neq 1$ or $y \neq 1$. If $x \neq 1$ and $y \neq 1$ then we have* $\lambda(yx^m \cdot y^n x) \geq (m-1)\lambda(x) + (n-1)\lambda(y) + 4$.

Proof. Obviously cancellation persists if $x = 1$ and $y = 1$, and dies if $x \neq 1 = y$ or $x = 1 \neq y$. Suppose, then, that $x \neq 1$ and $y \neq 1$.

If cancellation dies in $x \cdot y$ it dies in $yx^m \cdot y^n x$, and we have

$$\begin{aligned}\lambda(yx^m \cdot y^n x) &= m\lambda(x) + n\lambda(y) + \lambda(x \cdot y) \\ &\geq (m-1)\lambda(x) + (n-1)\lambda(y) + 4\end{aligned}$$

since $\lambda(x), \lambda(y) \geq 1$ and $\lambda(x \cdot y) \geq 2$.

If cancellation persists in $x \cdot y$ then $\lambda(x) \neq \lambda(y)$, else $x = \bar{y}$ and yx cannot be in F . Without loss of generality we may assume $\lambda(x) > \lambda(y)$, since the case $\lambda(y) > \lambda(x)$ is symmetrically treated.

Transcribing $x \to x_1\bar{y}$, where $x_1 \neq 1$, we have

$$yx^m \cdot y^n x = y(x_1\bar{y})^{m-2}x_1\bar{y}x_1 \cdot y^{n-1}x_1\bar{y}$$

and now by Theorem 15 cancellation dies in this expression, and its length is at least

$$(m-1)\lambda(x_1) + m\lambda(y) + (n-2)\lambda(y) + 4 = (m-1)\lambda(x) + (n-1)\lambda(y) + 4 \ . \qquad //$$

The bound given by the above theorem is attained for $x = a(ba^s)^{n-1}$, $y = \bar{a}^s\bar{b}$, where a, b are letters and s is an arbitrary positive integer.

THEOREM 18. *Let m, n be positive integers. Cancellation dies in the expression $\bar{y}x^m \cdot y^n\bar{x} \cdot \bar{x}$ if and only if x, y are not coradical, and then*

$$\lambda(\bar{y}x^m \cdot y^n\bar{x} \cdot \bar{x}) \geq m\lambda(x) + (n-1)\lambda(y) + 4 \ .$$

Proof. Suppose first that x, y are coradical. Then $\bar{x}$, y are strictly coradical, by Theorem 4, since $y\bar{x}$ appears in the above expression. Therefore we have $\bar{x} = z^r$, $y = z^s$, where z is a common root and r, s are non-negative integers, not both zero. (The case $r = s = 0$ may be subsumed under $z = 1$.) The expression now becomes

$$\bar{z}^{mr+s} \cdot z^{ns+r} \cdot z^r$$

and it is easily verified that cancellation persists.

Suppose now that x, y are not coradical. If x is cyclically reduced, the expression becomes

$$\bar{y}x^m \cdot y^n \bar{x}\bar{x}$$

and by Theorem 13 cancellation dies, and the length of the expression has the required lower bound. If x is not cyclically reduced, we observe that $\bar{y}x$ is reduced and therefore $x \cdot y = xy$, so, since $x \neq 1$, $y \neq 1$, the expression becomes

$$\bar{y}x^m y^n \bar{x} \cdot \bar{x}$$

and by Corollary 11.1 the length of this is at least $(m+1)\lambda(x) + (n+1)\lambda(y) + 1$, and since $\lambda(x) \geq 1$ and $\lambda(y) \geq 1$ this is at least $m\lambda(x) + (n-1)\lambda(y) + 4$. //

THEOREM 19. *Let* m *be a positive integer. Cancellation dies in the expression* $yx^m \cdot \bar{y}\bar{x}y \cdot \bar{x}\bar{y}$ *if and only if* $y \neq 1$ *, and in this case we have* $\lambda(yx^m \cdot \bar{y}\bar{x}y \cdot \bar{x}\bar{y}) \geq m\lambda(x) + 4$.

Proof. If $y = 1$ the expression becomes $x^m \cdot \bar{x} \cdot \bar{x}$, and it is clear that cancellation persists.

Suppose now that $y \neq 1$. By Theorem 6, since $\bar{y}\bar{x}y$ appears in the above expression, the words x, y are not coradical.

We may transcribe $y \to y_1 x^s$, where the integer s is chosen as large as possible. The integer s is well-defined, since $\lambda(x) \geq 1$ and $\lambda(y)$ is finite; if x is not a final subword of y then $s = 0$ and $y = y_1$. In any case x, y_1 are not coradical, by Corollary 4.1, and we have

$$yx^m \cdot \bar{y}\bar{x}y \cdot \bar{x}\bar{y} = y_1 x^m \cdot \bar{y}_1 \bar{x} y_1 \cdot \bar{x}\bar{y}_1 \ .$$

If cancellation dies in $x \cdot \bar{y}_1$ then *a fortiori* cancellation dies in this expression, and we have

$$\begin{aligned}\lambda(yx^m \cdot \bar{y}\bar{x}y \cdot \bar{x}\bar{y}) &= \lambda(y_1 x^{m-1}) + \lambda(x) + \lambda(y_1) + 2\lambda(x \cdot \bar{y}_1) \\ &= m\lambda(x) + 2\lambda(y_1) + 2\lambda(x \cdot \bar{y}_1) \\ &\geq m\lambda(x) + 6 \ ,\end{aligned}$$

since $\lambda(y_1) \geq 1$ and $\lambda(x \cdot \bar{y}_1) \geq 2$.

If cancellation persists in $x \cdot \bar{y}_1$, then we may transcribe $x \to x_1 y_1$. The other alternative, namely that we may transcribe $y_1 \to y_2 x$ for some y_2 , is ruled out because this would imply that s was not largest possible, contrary to its definition. We have therefore

$$yx^m \cdot \bar{y}\bar{x}y \cdot \bar{x}\bar{y} = (y_1 x_1)^{m-1} y_1 x_1 \cdot \bar{y}_1 \bar{x}_1 \cdot \bar{x}_1 \bar{y}_1$$

where x_1, y_1 are not coradical; Theorem 18 now shows that cancellation dies in this expression, and we have

$$\begin{aligned}\lambda(yx^m \cdot \bar{y}\bar{x}y \cdot \bar{x}\bar{y}) &= (m-1)\lambda(y_1 x_1) + \lambda(y_1 x_1 \cdot \bar{y}_1 \bar{x}_1 \cdot \bar{x}_1) + \lambda(y_1) \\ &\geq (m-1)\lambda(x) + \lambda(x_1) + 4 + \lambda(y_1) \\ &= m\lambda(x) + 4 \ . \qquad //\end{aligned}$$

THEOREM 20. *Let m, n be positive integers. Cancellation dies in the expression $\bar{y}(\bar{z}\bar{x})^m \cdot (yz)^n x$ if and only if xz, yz are not coradical, and in this case we have*

$$\lambda\left(\bar{y}(\bar{z}\bar{x})^m \cdot (yz)^n x\right) \geq (m-1)\lambda(xz) + (n-1)\lambda(yz) + 4 \ .$$

Proof. If $z = 1$ this is given by Theorem 13, so we may assume $z \neq 1$.

If $x = 1$, the expression is $\bar{y}\bar{z}^m \, yz(yz)^{n-1}$, in which, by Theorem 13, cancellation dies if and only if y, z are not coradical, in which case the length is at least $(m-1)\lambda(z) + 4 + (n-1)\lambda(yz)$, that is, $(m-1)\lambda(xz) + (n-1)\lambda(yz) + 4$. But since yz is reduced, y and z are coradical if and only if they are strictly coradical, by Theorem 4, and by Theorem 5 this is equivalent to coradicality of z and yz , that is, of xz and yz . A symmetrical argument disposes of the case $y = 1$, so we may assume $x \neq 1 \neq y$.

But now $(yz)^n xz$ and $(xz)^m yz$ are reduced, and by Theorem 4,

xz, yz are coradical iff they are strictly coradical, in which case cancellation persists in $\bar{z}\bar{y}(\bar{z}\bar{x})^m \cdot (yz)^n xz$, and *a fortiori* cancellation persists in $\bar{y}(\bar{z}\bar{x})^m \cdot (yz)^n x$.

Suppose now that xz, yz are not coradical.

If cancellation dies in $\bar{x} \cdot y$, the length of the above expression is

$$\begin{aligned}\lambda\left(\bar{y}(\bar{z}\bar{x})^{m-1}\bar{z}\right) + \lambda(\bar{x} \cdot y) + \lambda\left(z(yz)^{m-1}x\right) &\geq m\lambda(x) + m\lambda(y) + 2m\lambda(z) + \lambda(\bar{x} \cdot y) \\ &\geq (m-1)\lambda(xz) + (n-1)\lambda(yz) + 4\end{aligned}$$

since $\lambda(x), \lambda(y) \geq 1$ and $\lambda(\bar{x} \cdot y) \geq 2$.

If cancellation persists in $\bar{x} \cdot y$, we may assume without loss of generality that $\lambda(y) > \lambda(x)$ (as usual we have $\lambda(y) \neq \lambda(x)$, and a symmetry argument disposes of the case $\lambda(y) < \lambda(x)$). Transcribing $y \to xy_1$, we obtain the expression

$$\bar{y}_1(\bar{x}\bar{z})^m \cdot y_1 zx(y_1 zx)^{n-1} ,$$

where xz, $xy_1 z$ are not coradical. Now, since zx is reduced, Corollary 4.2 implies that y_1, zx are not coradical. Therefore cancellation dies in $\bar{y}_1(\bar{x}\bar{z})^m \cdot y_1 zx$, by Theorem 13, and the above expression has length at least

$$(m-1)\lambda(zx) + 4 + (n-1)\lambda(y_1 zx) = (m-1)\lambda(xz) + (n-1)\lambda(yz) + 4 . \qquad //$$

THEOREM 21. *Cancellation dies in the expression* $yx \cdot \bar{y}\bar{x} \cdot yx \cdot \bar{y}\bar{x}$ *if and only if* x, y *are not coradical, and in this case we have*

$$\lambda(yx \cdot \bar{y}\bar{x} \cdot yx \cdot \bar{y}\bar{x}) \geq \lambda(yx \cdot \bar{y}\bar{x}) + 4 .$$

Proof. If x, y are coradical then they are strictly coradical (since yx is reduced), and obviously cancellation persists.

Suppose, then, that x, y are not coradical. We have $\lambda(x), \lambda(y) \geq 1$, and clearly if $\lambda(x) = 1$ and $\lambda(y) = 1$ we have $x \cdot \bar{y} = x\bar{y}$ and $\bar{x} \cdot y = \bar{x}y$, so that the length of the expression is precisely 8, and $\lambda(yx \cdot \bar{y}\bar{x}) = 4$.

Now we consider the general case, arguing by induction on $\lambda(yx)$.

If cancellation dies in $x \cdot \bar{y}$ we transcribe $(x, y) \to (x_1 z, y_1 z)$, where $x_1 \cdot \bar{y} = x_1\bar{y}_1$, $x_1 \neq 1$, $y_1 \neq 1$. The expression becomes

$$y_1 zx_1(\bar{y}_1\bar{z}\bar{x}_1 \cdot y_1 zx_1)\bar{y}_1\bar{z}\bar{x}_1 ,$$

in which, by Theorem 20, cancellation dies in the bracketed portion and the length of this portion is at least 4. The length of the expression is thus at least $4 + \lambda(y_1 z x_1) + \lambda(\bar{y}_1 \bar{z}\bar{x}_1)$. But in the present case we have also

$$yx \cdot \bar{y}\bar{x} = y_1 z x_1 \bar{y}_1 \bar{z}\bar{x}_1 ,$$

and therefore $\lambda(yx \cdot \bar{y}\bar{x} \cdot yx \cdot \bar{y}\bar{x}) \geq \lambda(yx \cdot \bar{y}\bar{x}) + 4$.

If cancellation dies in $\bar{x} \cdot y$ we transcribe $(x, y) \to (zx_1, zy_1)$, where $\bar{x}_1 \cdot y_1 = \bar{x}_1 y_1$, $x_1 \neq 1$, $y_1 \neq 1$. The expression becomes

$$z(y_1 z x_1 \cdot \bar{y}_1 \bar{z}\bar{x}_1)(y_1 z x_1 \cdot \bar{y}_1 \bar{z}\bar{x}_1)\bar{z}$$

in which, by Theorem 20, cancellation dies in the bracketed portions, each of which has length at least 4 . Now we have also

$$yx \cdot \bar{y}\bar{x} = z(y_1 z x_1 \cdot \bar{y}_1 \bar{z}\bar{x}_1)\bar{z} ,$$

so that in this case again the inequality holds as claimed.

Finally, if cancellation persists in $x \cdot \bar{y}$ and in $\bar{x} \cdot y$, by Theorem 9 we may transcribe $(x, y) \to \big((x_1 y_1)^m x_1, (x_1 y_1)^n x_1\big)$ where x_1, y_1 are not coradical and where m, n are non-negative integers differing by 1 . Now, if $m = n + 1$ the expression becomes

$$(x_1 y_1)^n x_1 x_1 y_1 \cdot \bar{x}_1 \bar{y}_1 \cdot x_1 y_1 \cdot \bar{x}_1 (\bar{y}_1 \bar{x}_1)^m$$
$$= (x_1 y_1)^n x_1 (x_1 y_1 \cdot \bar{x}_1 \bar{y}_1 \cdot x_1 y_1 \cdot \bar{x}_1 \bar{y}_1) \bar{x}_1 (\bar{y}_1 \bar{x}_1)^n$$

so, by the inductive hypothesis, cancellation dies in the bracketed portion, and this portion has length at least $\lambda(x_1 y_1 \cdot \bar{x}_1 \bar{y}_1) + 4$. In this case we have also

$$yx \cdot \bar{y}\bar{x} = (x_1 y_1)^n x_1 (x_1 y_1 \cdot \bar{x}_1 \bar{y}_1) \bar{x}_1 (\bar{y}_1 \bar{x}_1)^n$$

so that the inequality holds as claimed. On the other hand if $n = m + 1$ the expression becomes

$$(x_1 y_1)^m x_1 (y_1 x_1 \cdot \bar{y}_1 \bar{x}_1 \cdot y_1 x_1 \cdot \bar{y}_1 \bar{x}_1) \bar{x}_1 (\bar{y}_1 \bar{x}_1)^m ,$$

we have $yx \cdot \bar{y}\bar{x} = (x_1 y_1)^m x_1 (y_1 x_1 \cdot \bar{y}_1 \bar{x}_1) \bar{x}_1 (\bar{y}_1 \bar{x}_1)^m$, and again the required inequality follows. //

THEOREM 22. *If x, y are not coradical, and if xy and yx are reduced words, we have*

$$\lambda(yx \cdot \bar{y}\bar{x}) + \lambda(\bar{y}\bar{x} \cdot yx) \geq 2\lambda(yx) + 4 .$$

Proof. If $\lambda(x) = 1$ and $\lambda(y) = 1$ then no non-trivial cancellation can take place, and the inequality becomes an equality.

We now consider the general case, arguing by induction on $\lambda(xy)$.

If cancellation dies in $x \cdot \bar{y}$, we transcribe $(x, y) \to (x_1 z, y_1 z)$, where $x_1 \neq 1$, $y_1 \neq 1$, and $x_1 \cdot \bar{y}_1 = x_1 \bar{y}_1$. Now we have

$$yx\cdot\bar{y}\bar{x} = y_1 z x_1 \bar{y}_1 \bar{z}\bar{x}_1 \ ,$$

and

$$\bar{y}\bar{x}\cdot yx = \bar{z}(\bar{y}_1\bar{z}\bar{x}_1\cdot y_1 z x_1)z \ ,$$

and by Theorem 20 cancellation dies in the bracketed portion, which has length at least 4 . It is now clear that the inequality holds in this case. A symmetrical argument establishes the inequality if cancellation dies in $\bar{x}\cdot y$.

Suppose now that cancellation persists in $x\cdot\bar{y}$ and in $\bar{x}\cdot y$. By Theorem 9 we may transcribe $(x, y) \to \big((x_1y_1)^m x_1, (x_1y_1)^n x_1\big)$, where x_1, y_1 are not coradical and m, n are non-negative integers differing by 1 . If $m = n + 1$ we have

$$yx\cdot\bar{y}\bar{x} = (x_1y_1)^n x_1(y_1x_1\cdot\bar{y}_1\bar{x}_1)\bar{x}_1(\bar{y}_1\bar{x}_1)^n \ ,$$

$$\bar{y}\bar{x}\cdot yx = (\bar{x}_1\bar{y}_1)^n\bar{x}_1(\bar{y}_1\bar{x}_1\cdot y_1x_1)x_1(y_1x_1)^n \ ,$$

and by the inductive hypothesis we have

$$\begin{aligned}\lambda(yx\cdot\bar{y}\bar{x}) + \lambda(\bar{y}\bar{x}\cdot yx) &= 4\lambda\big((x_1y_1)^n x_1\big) + \lambda(y_1x_1\cdot\bar{y}_1\bar{x}_1) + \lambda(\bar{y}_1\bar{x}_1\cdot y_1x_1)\\ &\geq 2\lambda\big((x_1y_1)^m x_1\big) + 2\lambda\big((x_1y_1)^n x_1\big) + 4\\ &= 2\lambda(xy) + 4 \ .\end{aligned}$$

A similar argument establishes the inequality if $n = m + 1$. //

THEOREM 23. *Let m, n, r be positive integers. If x, y are not coradical, and if $yx, \bar{y}x, y\bar{x}$ are reduced words, then we have*

$$\lambda(\bar{y}x^m\cdot yx) + \lambda(\bar{y}x^n\cdot y^r\bar{x}) \geq (m+n)\lambda(x) + (r-1)\lambda(y) + 4 \ .$$

Proof. Theorems 13 and 14 assure us that cancellation dies in the expressions indicated.

If cancellation dies in $x\cdot y$, then the total length of the expressions is at least

$$(m+n)\lambda(x) + (r+1)\lambda(y) + 4 \ ,$$

since $\lambda(x\cdot y) \geq 2$, so that in this case the inequality holds. In particular this is true if $\lambda(x) = 1$ and $\lambda(y) = 1$, so that x, y are independent letters. We may therefore argue by induction on $\lambda(yx)$ in dealing with the remaining cases.

If cancellation persists in $x\cdot y$, we may transcribe $x \to x_1\bar{y}$ or $y \to \bar{x}y_1$. The transcription $y \to \bar{x}y_1$ gives

$$\bar{y}x^m\cdot yx = \bar{y}_1x^m\cdot y_1x$$

and

$$\bar{y}x^n \cdot y^r\bar{x} = \bar{y}_1 x^n \cdot y_1\bar{x}(y_1\bar{x})^{r-1}$$

where x, y_1 are not coradical, and the inductive hypothesis implies that the total length of the two expressions is at least $(m+n)\lambda(x) + 4 + (r-1)\lambda(y_1\bar{x})$, as required. The transcription $x \to x_1\bar{y}$, on the other hand, gives

$$\bar{y}x^m \cdot yx = (\bar{y}x_1)^{m-1}\bar{y}x_1 \cdot x_1\bar{y}$$

and

$$\bar{y}x^n \cdot y^r\bar{x} = (\bar{y}x_1)^{n-1}\bar{y}x_1 \cdot y^r\bar{x}_1$$

where x_1, y are not coradical. Now, if x_1 is cyclically reduced, so that no non-trivial cancellation is possible in the first expression, then the total length is at least $m\lambda(\bar{y}x_1) + \lambda(x_1\bar{y}) + (n-1)\lambda(\bar{y}x_1) + (r-1)\lambda(y) + 4$, by Theorem 13 - that is, at least $(m+n)\lambda(x) + (r-1)\lambda(y) + 4$. If x_1 is not cyclically reduced, then no non-trivial cancellation is possible in the second expression, since $\bar{y}x_1$ is a reduced word. The total length in this case is therefore at least

$$(m-1)\lambda(\bar{y}x_1) + 2\lambda(y) + \lambda(x_1) + 1 + n\lambda(\bar{y}x_1) + (r-1)\lambda(y) + \lambda(y\bar{x}_1) \; ,$$

by Corollary 11.1, that is, at least

$$(m+n)\lambda(x) + (r-1)\lambda(y) + 2\lambda(y) + \lambda(x_1) + 1 \geq (m+n)\lambda(x) + (r-1)\lambda(y) + 4 \; . \qquad //$$

THEOREM 24. *Let* m, n *be positive integers. Then cancellation dies in the expression*

$$x(zy)^m \cdot (\bar{x}\bar{z})^n \bar{y}x \cdot \bar{y}\bar{z}\bar{x}$$

if and only if zx, zy *are not coradical, and in this case the length of the expression is at least*

$$m\lambda(zy) + (n-1)\lambda(zx) + 4 \; .$$

Proof. Suppose first that zx, zy are coradical. If $x = 1$ then z, zy are coradical, and therefore z, y are strictly coradical, by Theorem 5. Now, the expression in this case becomes

$$(zy)^m \cdot \bar{z}^n \bar{y} \cdot \bar{y}\bar{z}$$

and it is clear that cancellation persists. If $x \neq 1$, we see that $zx(zy)^m$ and $\bar{y}\bar{z}\bar{x}\bar{z}$ are reduced, It is now clear that cancellation persists in the expression

$$zx(zy)^m \cdot (\bar{x}\bar{z})^n \bar{y}x \cdot \overline{yzxz} = zx(zy)^m \cdot (\bar{x}\bar{z})^n \bar{y} \cdot \bar{z} \cdot z \cdot x \cdot \overline{yzxz}$$
$$= zx(zy)^m \cdot (\bar{x}\bar{z})^n \cdot \bar{y}\bar{z} \cdot zx \cdot \overline{yzxz}$$

since zx, zy are strictly coradical, and therefore *a fortiori* cancellation persists in the expression $x(zy)^m \cdot (\bar{x}\bar{z})^n \bar{y}x \cdot \overline{yzx}$ as claimed.

Suppose now that zx, zy are not coradical.

If cancellation dies in $x \cdot \bar{y}$ then the expression is

$$x(zy)^{m-1} z(y \cdot \bar{x})\bar{z}(\bar{x}\bar{z})^{n-1} \bar{y}(x \cdot \bar{y})\bar{z}\bar{x}$$

and its length is

$$m\lambda(zy) + (n+1)\lambda(zx) + 2\lambda(x \cdot \bar{y}) \geq m\lambda(zy) + (n-1)\lambda(zx) + 6$$

since $\lambda(zx) \geq 1$ and $\lambda(x \cdot \bar{y}) \geq 2$.

If cancellation persists in $x \cdot \bar{y}$ then we may transcribe either $x \to x_1 y$ or $y \to y_1 x$. The transcription $y \to y_1 x$ gives

$$(xzy_1)^{m-1} \left(xzy_1 \cdot (\bar{z}\bar{x})^n \bar{y}_1 \cdot \bar{y}_1\right) \bar{z}\bar{x}$$

where zx, $zy_1 x$ are not coradical, and hence y_1, xz are not coradical. But now Theorem 18 shows that cancellation dies in this expression, and the length of the expression is at least

$$(m-1)\lambda(xzy_1) + \lambda(y_1) + (n-1)\lambda(xz) + 4 + \lambda(xz)$$
$$= m\lambda(zy) + (n-1)\lambda(zx) + 4 \ .$$

On the other hand, the transcription $x \to x_1 y$ gives

$$x_1(yz)^m \cdot (\bar{x}_1 \bar{z}\bar{y})^n x_1 \cdot \overline{zyx_1}$$

where $zx_1 y$, zy are not coradical, and hence x_1, yz are not coradical. If $n = 1$ then by Theorem 19 cancellation dies in this expression, and its length is at least $m\lambda(yz) + 4$, that is, at least $m\lambda(zy) + (n-1)\lambda(zx) + 4$. But if $n > 1$ then the expression is

$$\left(x_1(yz)^m \cdot \bar{x}_1 \bar{z}\bar{y}\right)(\bar{x}_1 \bar{z}\bar{y})^{n-2} \bar{x}_1 (\bar{z}\bar{y}x_1 \cdot \overline{zyx_1})$$

and now cancellation dies in the bracketed portions, and by Theorem 23 the total length is at least

$$(m+1)\lambda(yz) + 4 + (n-2)\lambda(yzx_1) + \lambda(x_1)$$
$$= m\lambda(zy) + (n-1)\lambda(zx) + 4 \ . \qquad //$$

Setting $z = 1$, we obtain a generalization of Theorem 19.

COROLLARY 24.1. *Let* m, n *be positive integers. Then cancellation dies in the expression* $xy^m \cdot \bar{x}^n \bar{y}x \cdot \bar{y}\bar{x}$ *if and only if* $x \neq 1$ *, and in this case we have*

$$\lambda(xy^m\cdot\bar{x}^n\bar{y}x\cdot\bar{y}\bar{x}) \geq m\lambda(y) + (n-1)\lambda(x) + 4 \;.$$

Proof. By the above theorem, cancellation dies in the expression, and its length satisfies the required bound, if and only if x, y are not coradical. But $\bar{x}\bar{y}x$ appears as a reduced word, so, by Theorem 6, x and y are coradical if and only if $x = 1$. //

THEOREM 25. *Let m be a positive integer. Then cancellation dies in the expression*

$$\bar{y}(zx)^m\cdot yxz$$

if and only if $zx \neq 1$, in which case we have

$$\lambda\left(\bar{y}(zx)^m\cdot yxz\right) \geq m\lambda(zx) + 2 \;.$$

Proof. Clearly $zx = 1$ iff $xz = 1$, and in this case cancellation persists.

Suppose now that $zx \neq 1$, and hence that $xz \neq 1$. If $x = 1$ or $z = 1$ our assertion is given by Theorem 14, so we may make the stronger assumption: $x \neq 1$ and $z \neq 1$.

If $y = 1$, the expression becomes $(zx)^{m-1}zx\cdot xz$, and by Corollary 11.1 the length is at least $(m-1)\lambda(zx) + \lambda(x) + 1 + 2\lambda(z)$, hence at least $m\lambda(zx) + 2$, since $\lambda(z) \geq 1$.

If $y \neq 1$, we consider separately the cases in which cancellation dies or persists in $x\cdot y$.

If cancellation dies in $x\cdot y$, the expression is

$$\bar{y}(zx)^{m-1}z(x\cdot y)xz \;,$$

and its length is

$$m\lambda(zx) + \lambda(y) + \lambda(z) + \lambda(x\cdot y) \geq m\lambda(zx) + 4 \;.$$

If cancellation persists in $x\cdot y$, we may transcribe either $x \to x_1\bar{y}$ or $y \to \bar{x}y_1$. The transcription $x \to x_1\bar{y}$ gives

$$(\bar{y}zx_1)^{m-1}\bar{y}zx_1\cdot x_1\bar{y}z \;,$$

and since yx was reduced so is $yx_1\bar{y}$, which implies by Theorem 6 that x_1, y are not coradical and in particular we have $x_1 \neq 1$, so that the length of our expression is at least

$$\begin{aligned}(m-1)\lambda(\bar{y}zx_1) + 2\lambda(\bar{y}z) + \lambda(x_1) + 1 &\geq m\lambda(\bar{y}zx_1) + 3 \\ &= m\lambda(zx) + 3 \;.\end{aligned}$$

On the other hand, the transcription $y \to \bar{x}y_1$ gives

$$\bar{y}_1(xz)^m\cdot y_1xz$$

and now by Theorem 14 cancellation dies, and the length of the expression is at least

$$m\lambda(xz) + 2 = m\lambda(zx) + 2 . \qquad //$$

THEOREM 26. *Let* m, n, r *be positive integers. If* zx, zy *are not coradical, and if* yzx, $\bar{y}xz$, xzy *are reduced words, then we have*

$$\lambda\{y(zx)^m\cdot\bar{y}xz\} + \lambda\{y(zx)^n\cdot(\bar{y}\bar{z})^r\bar{x}\} \geq (m+n)\lambda(zx) + (r-1)\lambda(zy) + 4 .$$

Proof. Theorem 25 and Theorem 20 assure us that cancellation dies in the two expressions indicated.

If cancellation dies in $x\cdot\bar{y}$, the total length of the two expressions is at least

$$(m+n)\lambda(zx) + (r+1)\lambda(zy) + 4 ,$$

as is easily verified.

If cancellation persists in $x\cdot\bar{y}$ then we may transcribe $y \rightarrow y_1x$ or $x \rightarrow x_1y$. The transcription $y \rightarrow y_1x$ gives

$$y(zx)^m\cdot\bar{y}xz = y_1(xz)^m\cdot\bar{y}_1xz$$

and

$$y(zx)^n\cdot(\bar{y}\bar{z})^r\bar{x} = y_1(xz)^n\cdot\bar{y}_1\bar{z}\bar{x}(\bar{y}_1\bar{z}\bar{x})^{r-1}$$

where zx, zy_1x are not coradical, and hence xz, y_1 are not coradical. By Theorem 23, cancellation dies in each of the expressions, and the total length is $(m+n)\lambda(xz) + 4 + (r-1)\lambda(xzy_1)$, that is, $(m+n)\lambda(zx) + (r-1)\lambda(zy) + 4$. The transcription $x \rightarrow x_1y$ gives

$$y(zx)^m\cdot\bar{y}xz = (yzx_1)^{m-1}yzx_1\cdot x_1yz$$

and

$$y(zx)^n\cdot(\bar{y}\bar{z})^r\bar{x} = (yzx_1)^{n-1}yzx_1\cdot(\bar{z}\bar{y})^r\bar{x}_1$$

where zx_1y, zy are not coradical, and hence x_1, yz are not coradical. Now, if x_1 is cyclically reduced, then non-trivial cancellation can take place only in the second expression, so that the total length is at least

$$(m+n-1)\lambda(yzx_1) + \lambda(x_1yz) + (r-1)\lambda(yz) + 4$$
$$= (m+n)\lambda(zx) + (r-1)\lambda(zy) + 4 ,$$

by Theorem 13. On the other hand, if x_1 is not cyclically reduced then no non-trivial cancellation can take place in the second expression, since yzx_1 appears as a reduced word, so that $x_1 \cdot \bar{z}\bar{y} = x_1\bar{z}\bar{y}$. By Corollary 11.1, therefore, the total length of the two expressions is at least

$$\begin{aligned}(m+n-1)\lambda(yzx_1) + \lambda(x_1) + 1 + \lambda(yz) + \lambda\left((\bar{z}\bar{y})^r\bar{x}_1\right) \\ = (m+n)\lambda(zx) + (r-1)\lambda(zy) + 2\lambda(x_1) + 1 + 2\lambda(yz) \\ > (m+n)\lambda(zx) + (r-1)\lambda(zy) + 4 . \qquad //\end{aligned}$$

4. Coradicality and commuting words

It will be plain to the reader that the question of whether two elements of F commute is very closely related to the question of whether they are coradical. Certainly, if x and y are coradical, then $x \cdot y = y \cdot x$ (a semigroup power $x = z^m$ is also a power of z in terms of group multiplication, and similarly for y); differently put, if x and y do not commute in F then they are not coradical. But the converse is true only "up to an inner automorphism", in the following sense.

THEOREM 27. *If x, y are words in F such that $x \cdot y = y \cdot x$, then there is a word z in F such that $x = zx_0\bar{z}$ and $y = zy_0\bar{z}$, where x_0, y_0 are coradical words in F .*

Proof. Suppose, first, that cancellation persists in $x \cdot y$. Without loss of generality we may assume $\lambda(x) \geq \lambda(y)$, so that cancellation proceeds via the transcription $x \to x_1\bar{y}$, where x_1 is some word in F . The equation $x \cdot y = y \cdot x$ now becomes

$$x_1 = y \cdot x_1\bar{y} ,$$

and multiplying on the left by $\bar{y}$ gives

$$\bar{y} \cdot x_1 = x_1\bar{y} .$$

Now we have $\lambda(\bar{y} \cdot x_1) = \lambda(x_1\bar{y}) = \lambda(x_1) + \lambda(y)$, so that no non-trivial cancellation is possible in $\bar{y} \cdot x_1$. That is, we have $\bar{y}x_1 = x_1\bar{y}$, so that x_1 and $\bar{y}$ are strictly coradical, by Theorem 1, and therefore x, y are coradical. In this case, then, we may put $z = 1$, $x_0 = x$, $y_0 = y$.

Suppose now that cancellation dies in $x \cdot y$. Accordingly we may transcribe $(x, y) \to (x_1\bar{u}, uy_1)$, where $x_1 \neq 1$, $y_1 \neq 1$, and $x_1 \cdot y_1 = x_1y_1$. The equation $x \cdot y = y \cdot x$ becomes

$$x_1y_1 = uy_1 \cdot x_1\bar{u} .$$

We now ask whether or not cancellation dies in $y_1 \cdot x_1$.

If cancellation dies in $y_1 \cdot x_1$ then we may transcribe $(x_1, y_1) \to (vx_2, y_2\bar{v})$, where $x_2 \neq 1$, $y_2 \neq 1$, and $y_2 \cdot x_2 = y_2x_2$. We have now

$$vx_2y_2\bar{v} = uy_2x_2\bar{u} ,$$

and now, equating lengths, we find $\lambda(u) = \lambda(v)$, and by Corollary 1.1 we have $u = v$, and therefore $x_2y_2 = y_2x_2$. By Theorem 2, x_2 and y_2 are strictly coradical, and our assertion holds with $x_0 = x_2$, $y_0 = y_2$ and $z = u = v$.

If cancellation persists in $y_1 \cdot x_1$, then without loss of generality we may assume $\lambda(y_1) \geq \lambda(x_1)$; the case $\lambda(y_1) \leq \lambda(x_1)$ may be dealt with by a symmetrical argument. Transcribing $y_1 \to y_3\bar{x}_1$, we have, after cancellation,

$$x_1y_3\bar{x}_1 = uy_3 \cdot \bar{u} ,$$

where $x = x_1\bar{u}$ and $y = uy_3\bar{x}_1$. Clearly

$$\lambda(uy_3 \cdot \bar{u}) \leq \lambda(uy_3) + \lambda(\bar{u}) = 2\lambda(u) + \lambda(y_3) .$$

But we have also $2\lambda(x_1) + \lambda(y_3) = \lambda(x_1y_3\bar{x}_1) = \lambda(uy_3 \cdot \bar{u})$, and therefore $\lambda(x_1) \leq \lambda(u)$.

Multiplying both sides of our equation by u on the right, we have

$$x_1y_3\bar{x}_1 \cdot u = uy_3$$

and now there is a word w such that $u = x_1w$. For, if cancellation persists in $\bar{x}_1 \cdot u$ then by the inequality $\lambda(x_1) \leq \lambda(u)$ it does so via a transcription $u \to x_1w$ for some w, whereas if cancellation dies in $\bar{x}_1 \cdot u$ then x_1y_3, and *a fortiori* x_1, is an initial subword of uy_3, so by Theorem 1 and the same inequality $\lambda(x_1) \leq \lambda(u)$ we have again $u = x_1w$ for some w. Substituting for u, we have therefore

$$x_1y_3\bar{x}_1 \cdot x_1w = x_1wy_3$$

and, after cancelling,

$$x_1y_3 \cdot w = x_1wy_3 .$$

Since $\lambda(x_1y_3) + \lambda(w) = \lambda(x_1wy_3)$, no non-trivial cancellation is possible on the left side of this equation, so we have $x_1y_3w = x_1wy_3$, hence $y_3w = wy_3$, and by Theorem 2 the words w and y_3 are strictly coradical. But now $\bar{w}$ and wy_3 are coradical, and since $x = x_1\bar{u} = x_1\bar{w}\bar{x}_1$ and $y = uy_3\bar{x}_1 = x_1wy_3\bar{x}_1$ our assertion holds with $x_0 = \bar{w}$, $y_0 = wy_3$ and $z = x_1$. //

5. Cancellation flowcharts

In Section 3 we studied a miscellany of "excerpts", without needing to give a general definition of the term. But it will be clear that a metatheorem, embracing all "excerpts" and including as special cases all of our results from Theorem 11 to Theorem 26, would be impossibly clumsy, and not worth stating. We can, however, say what we mean by "excerpt", and describe a general, diagrammatic, method of working which simplifies the task of formulating and proving results such as our Theorems 11-26.

DEFINITION. An *excerpt* of an unreduced expression in several variables (taking values in F) is a sub-expression which satisfies the following conditions:

(a) the excerpt includes cancellation-points,

(b) no cancellation-point in the main expression appears immediately before or immediately after the excerpt,

(c) every variable that appears (with or without an overline) in the excerpt appears at least once in that portion of the excerpt preceding its first cancellation-point, and in that portion of the excerpt following its last cancellation-point,

(d) no proper sub-expression of the excerpt satisfies conditions (a), (b), (c);

(e) with the proviso that if two or more overlapping sub-expressions satisfy conditions (a), (b), (c), (d), then only the combined expression (the join of these sub-expressions in the lattice of all sub-expressions) is to be counted as an excerpt.

According to conditions (b) and (e), whether a sub-expression is an excerpt depends not only on the sub-expression but also on its context in the main expression. Thus when we speak of an expression such as $yx^m \cdot \bar{y}\bar{x}$ as being an excerpt, we really mean that there are expressions in which it appears as an excerpt in accordance with the above definition. This "*abus de langage*" presents no problems: an "excerpt", in this sense, is an expression which is an excerpt of itself.

To take some examples: the expression $yx^m \cdot \bar{y}\bar{x}y \cdot \bar{x}\bar{y}$ of Theorem 19 is an excerpt, and so are $yx^m \cdot \bar{y}\bar{x}$ and $\bar{x}y \cdot \bar{x}\bar{y}$, but not in this context -

they overlap here, so that according to condition (e) of the definition we must take the combined expression as an excerpt. The expression $yx^m \cdot \bar{y}\bar{x}^2 y \cdot \bar{x}\bar{y}$ is not one excerpt but two contiguous excerpts. The expression $zyx \cdot \bar{y}\bar{x}\bar{z}$ is not an excerpt, because it violates condition (d): the sub-expression $yx \cdot \bar{y}\bar{x}$ is an excerpt. The expression $\bar{y}x^m \cdot y^n \bar{x} \cdot \bar{x}$ of Theorem 18 is not an excerpt according to our definition: it owes its place in our list of expressions to the fact that, under the conditions of Theorem 18, at least one of the cancellation-points can be removed without cancellation - that is, the expression is "really" either $\bar{y}x^m \cdot y^n \bar{x}^2$ or $\bar{y}x^m y^n \bar{x} \cdot \bar{x}$, and the ambiguity is resolved only when we know whether or not x is cyclically reduced.

In lieu of our "metatheorem", we now state without proof a general (and somewhat vague) proposition; later, we shall give heuristic arguments to support it.

PROPOSITION. *Cancellation dies in every excerpt, provided that certain conditions are satisfied, which may vary from one excerpt to the next; each of these conditions states either that a certain reduced expression denotes a word in* F *not equal to* 1 , *or else that two such expressions denote words in* F *which are not coradical.*

Let us consider, for example, the excerpt $xzy \cdot \bar{z}\bar{x}zy$. Clearly if $y = 1$ cancellation persists, so we must impose at least the condition $y \neq 1$. The transcription $z \to zy$ gives $xzy \cdot \bar{z}\bar{x}zy^2$, from which we take the excerpt $xzy \cdot \bar{z}\bar{x}zy$, and since the denotation of y is unchanged we still have $y \neq 1$. (Recall that the transcription $z \to zy$ is really a transcription $z \to z_1 y$, followed by a relabelling $z_1 \to z$; the net result is to keep the symbol z but to change its denotation.) Since we are not concerned at the moment with finding the length of the expression, the only other transcription that interests us is $y \to yz$, which gives $xzy \cdot \bar{x}zyz$, an expression from which we take the excerpt $xzy \cdot \bar{x}zy$. Continuing in this way, we arrive at a set of excerpts, and transcriptions which "lead" from one excerpt to another, as summarized by the diagram (Figure 1).

The condition $y \neq 1$ becomes $yz \neq 1$ after the transcription $y \to yz$: in the diagram, we have entered this "image" of the original condition under the excerpt $xzy \cdot \bar{x}zy$ to which it applies, and similarly we have entered the relevant version of this condition under each of the excerpts in our diagram.

$$
\begin{tikzcd}
\begin{matrix} xzy\cdot\bar{z}\bar{x}zy \\ y\neq1 \end{matrix} \arrow[loop right, "{[z \to zy]}"] \arrow[d, "{[y \to yz]}"'] & \\
\begin{matrix} xzy\cdot\bar{x}zy \\ yz\neq1 \end{matrix} \arrow[d, "{[x \to xy]}"'] \arrow[dr, "{[y \to yx]}"] & \\
\begin{matrix} xyz\cdot\bar{x}zy \\ yz\neq1 \end{matrix} \arrow[u, bend right, "{[x \to x\bar{z}]}"'] \arrow[d, "{[z \to zx]}"'] & \begin{matrix} x(\bar{z}y\cdot\bar{z}y)x \\ yxz\neq1 \end{matrix} \\
\begin{matrix} xy(z\cdot z)xy \\ yzx\neq1 \end{matrix} &
\end{tikzcd}
$$

Figure 1.

The "end-products" shown in the diagram, namely the expressions $xyz\cdot zxy$ and $x\bar{z}y\cdot\bar{z}yx$, are of course not excerpts. In these instances we deviate from the general practice of taking excerpts, so as to allow for the respective possibilities $z = 1$ and $\bar{z}y = 1$. Clearly, if $yzx \neq 1$ and xyz, zxy are reduced expressions then either $z \neq 1$ or else $z = 1$ and $xy \neq 1$; in each case Corollary 11.1 assures us that cancellation dies in the expression $xyz\cdot zxy$. Similarly, the condition $yxz \neq 1$ ensures that cancellation dies in $x\bar{z}y\cdot\bar{z}yx$, because we must have either $\bar{x}y \neq 1$ or $x \neq 1$.

For each excerpt in the diagram we must check that the result of giving to one of the variables that appear beside the cancellation-point the value 1 is subsumed under the transcription which removes this variable by cancellation with the other. For example, in the excerpt $xzy\cdot\bar{x}zy$, the result of setting x equal to 1 (namely $zy\cdot zy$) is subsumed under the result of transcribing $y \to yx$ and so removing the "$\bar{x}$" by cancellation (namely $x\bar{z}y \cdot \bar{z}yx$). Where this is not the case, the "degenerate" transcription must either be handled separately or shown to be excluded by virtue of the conditions originally imposed. For example, in the original excerpt $xzy \cdot \bar{z}\bar{x}zy$ the "degenerate" transcription $y \to 1$ is not subsumed under the transcription $z \to zy$. (A transcription is "subsumed under" another if the result of performing the first transcription can be considered, possibly with some relabelling of variables, as a special case of the result obtained when the second transcription is performed.)

When we are using the diagram (or "flowchart", as we shall call it) as an heuristic tool, to study cancellation in an excerpt with unspecified "side-conditions", we may be led in this way to modify or strengthen the conditions on the original expression so as to ensure that cancellation does not persist.

Of course, if at any stage in our flowcharting we arrive at an expression and set of conditions for which we have already established non-persistence, we need go no further. Thus, if one has previously established the result given in our Theorem 14, the flowchart stops at $xzy \cdot \bar{x}zy$ (since $yz \neq 1$ is equivalent to $zy \neq 1$).

The appearance of the flowchart in Figure 1, as an interconnected collection of loops, is typical of cancellation flowcharts in general. (We call them *flowcharts* because of their resemblance to the computer programmer's flowcharts, and to distinguish them from the cancellation *diagrams* of Van Kampen and Lyndon.) The "end-expressions" arrived at in a cancellation flowchart are normally either elaborations of $x \cdot x$ $(x \neq 1)$, as in Figure 1, or elaborations of $\bar{y}x \cdot y\bar{x}$ (x, y not coradical), or else "ambiguous" forms such as the one studied in Theorem 18.

Intuitively, it is fairly easy to see why this must be so. Consider for example what happens in a typical subexpression $\ldots x \cdot y \ldots$, containing just one cancellation-point. Under the transcription $x \to x\bar{y}$, the "y" beside the cancellation-point disappears. If the variable x does not occur in the reduced portion between the cancellation-point shown and the next one to the right (by definition of an excerpt, there must be at least one such cancellation-point if x does not occur) then the number of symbols in this reduced portion is reduced by at least one (perhaps two, as in $\ldots x \cdot y \ldots \bar{y} \cdot \bar{x} \ldots$). If x does occur in this portion, we have either $\ldots x \cdot yy_1 \ldots y_n x \ldots$ or $\ldots x \cdot yy_1 \ldots y_n\bar{x} \ldots$, where the variables $y, y_1, \ldots, y_n$ are all different from x and $\bar{x}$. In the expression $\ldots x{\cdot}yy_1 \ldots y_n x \ldots$, the transcription $x \to x\bar{y}$ gives $\ldots x{\cdot}y_1 \ldots y_n x\bar{y} \ldots$, so that the number of symbols between the two occurrences of x is reduced by 1 , while in the expression $\ldots x{\cdot}yy_1 \ldots y_n\bar{x} \ldots$ the same transcription merely permutes cyclically the symbols $y, y_1, \ldots, y_n$ between the occurrences of x and $\bar{x}$ shown, so that the further sequence of transcriptions $x \to x\bar{y}_1, x \to x\bar{y}_2, \ldots, x \to x\bar{y}_n$ will bring us back to the original expression $\ldots x{\cdot}yy_1 \ldots y_n\bar{x} \ldots$ unless the parts of the expression to the left of "x" and to the right of "$\bar{x}$" have been changed in the process.

If x occurs twice in the same unreduced portion, for example as in

$$\ldots xz_1z_2 \ldots z_m x{\cdot}yy_1 \ldots ,$$

then a transcription $x \to x\bar{y}$ gives

$$\ldots x\bar{y}z_1z_2 \ldots z_m x{\cdot}y_1 \ldots$$

in which the two occurrences of x are separated by one symbol more than at first; we note also that before the transcription there may have been no occurrences of y or $\bar{y}$ in the reduced portion to the left of the cancellation-point. If instead we have

$$\ldots \bar{x}z_1z_2 \ldots z_m x{\cdot}yy_1 \ldots$$

then the transcription $x \to x\bar{y}$, while not increasing the number of symbols appearing between "$\bar{x}$" and "x" , will usually introduce an extra occurrence of y or $\bar{y}$ in the reduced portion to the left of the cancellation-point (there are exceptions: for example, in $\ldots \bar{y}{\cdot}\bar{x}z_1z_2 \ldots z_m x{\cdot}yy_1 \ldots$ and in $\ldots x{\cdot}yw_1w_2 \ldots w_k\bar{x}z_1z_2 \ldots z_m x{\cdot}yy_1 \ldots$ the number of symbols between the cancellation-points, and the number of "y"s or "$\bar{y}$"s , is unchanged by the transcription $x \to x\bar{y}$).

The evident tendency, then, is for an excerpt, when subjected to the appropriate transcriptions, either to resolve itself into one or more simpler excerpts or else to traverse periodically a finite number of forms; in the latter case the periodicity may be complex, the transitions among these forms possibly being represented by several interconnected loops in the cancellation flowchart. In an excerpt with more than one cancellation-point, the resolution into simpler excerpts may take place either when the portion between two cancellation-points is entirely removed by cancellation (so that the two cancellation-points "merge" into one, and may even be eliminated altogether), or else, at the opposite extreme, when the transcriptions have introduced so many new occurrences of variables between two cancellation-points that it becomes possible to divide the whole expression into two or more non-overlapping excerpts (and perhaps some reduced portions left over), with the two cancellation-points now segregated in separate excerpts.

When we construct a flowchart, we are of course not entitled to close a "loop" by referring back to a previous entry, unless both the expression in that entry and the associated conditions are produced by the transcription currently being considered. Obviously, if we are concerned with expressions in which cancellation dies, there must always be some condition imposed on the original expression - otherwise all the variables might be given the value 1 , thus ensuring that cancellation persists. A loop in the flowchart corresponds to a sub-expression of the form $x \cdot y_1 y_2 \ldots y_n \bar{x}$, where $y_1, y_2, \ldots, y_n$ are variables other than x or $\bar{x}$, and the relevant transcriptions are $x \to x\bar{y}_1, x \to x\bar{y}_2, \ldots, x \to x\bar{y}_n$. Unless we have $y_1 y_2 \ldots y_n = 1$, and hence $y_1 = 1, y_2 = 1, \ldots, y_n = 1$, it is clear that some non-trivial cancellation must take place as the loop is traversed; it follows, then, that a loop can be traversed only finitely many times - in the example, $y_1 y_2 \ldots y_n = 1$ really implies that the loop is not traversed at all, and that instead we are dealing with a "degenerate" transcription which cannot be part of any loop. Since an excerpt has finitely many cancellation-points, it can be subject only to finitely many different transcriptions. It follows that no entry in a flowchart lies on infinitely many loops, and hence that no part of a loop can be traversed infinitely often. In proving some assertion by means of a flowchart, we may therefore eliminate any path of our choice by assuming that it has been traversed as often as possible - of course, we must then take into account the composite of all the transcriptions that we assume to have taken place.

Wherever possible, we eliminate cancellation-points as we did in proving Theorem 18. As an example, we may take the expression $yx^n \cdot \bar{y}\bar{x}y \cdot \bar{x}\bar{y}$ of Theorem 19, with the condition $y \neq 1$. Using Theorem 6, we find that this condition is equivalent to the condition that x and y are not coradical. This condition remains valid for the new values of x and y resulting after either of the transcriptions $x \to xy$, $y \to yx$. In Figure 2, we present the completed flowchart, showing (in an obvious graphical convention) how the ambiguity of the expression $yx \cdot \bar{y}\bar{x} \cdot \bar{x}\bar{y}$ is resolved.

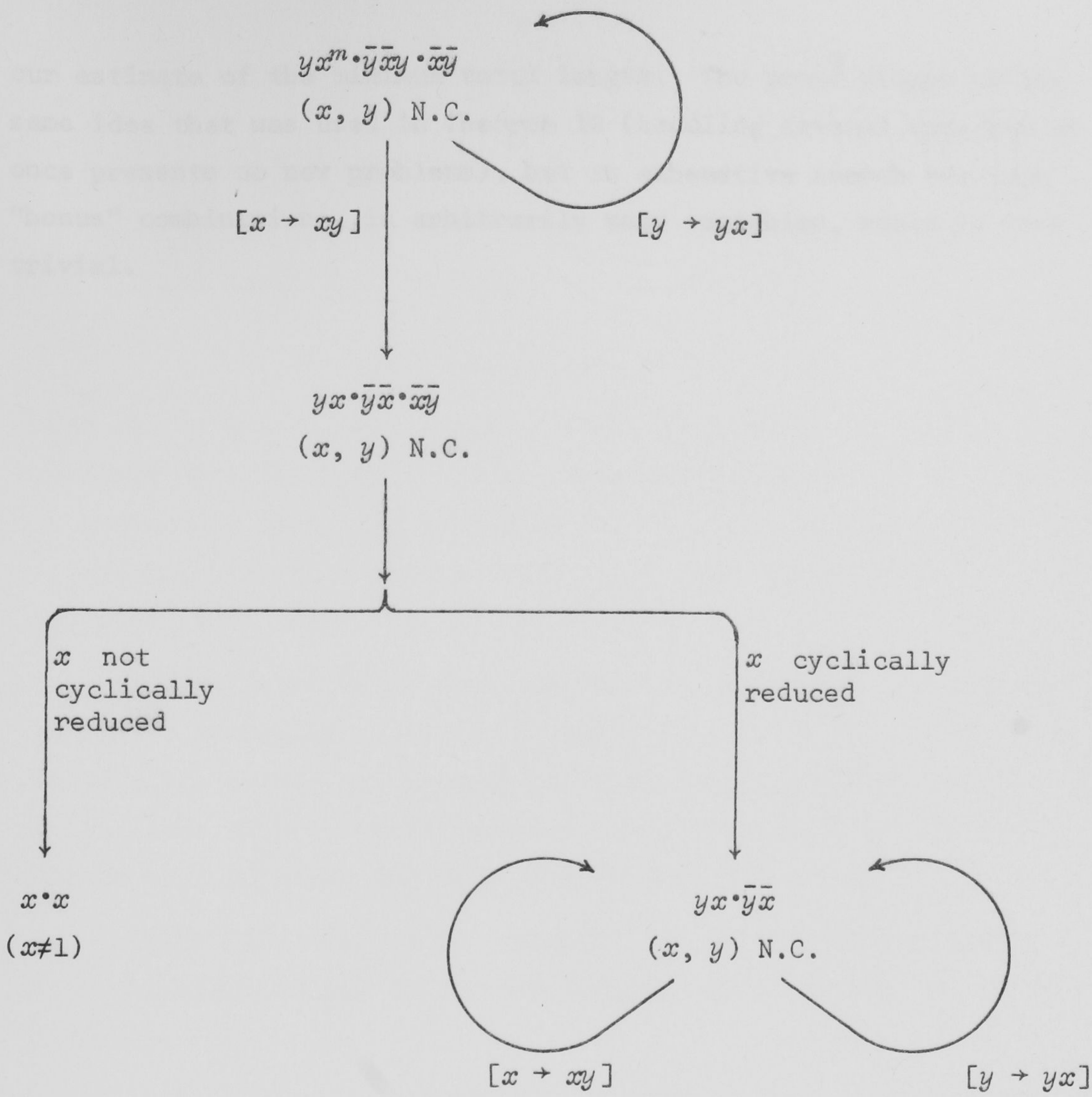

Figure 2.

When we use a flowchart as a basis for the proof of a statement about the length of some expression, we must, of course, take account of the lengths of any reduced portions that we discard on the way (for example, when after a transcription we take a proper excerpt of the resulting expression). Since the progress of this accounting depends very much on the expression being considered, we are forced to prove every such result individually. The flowcharting technique presented here thus emerges not as a general proof for all length-estimates but as a recipe for making a proof for each particular case. (We hesitate to call it an "algorithm", because combining the ingredients to give best results is not entirely a mechanical procedure.)

For example, the point of Theorem 22 is that, whereas for x, y not coradical we may expect to have $\lambda(yx\cdot\bar{y}\bar{x}) \geq 4$ and $\lambda(\bar{y}\bar{x}\cdot yx) \geq 4$, if these expressions occur together we get a bonus of $2\big(\lambda(xy)-2\big)$ in our estimate of the minimum total length. The proof hinges on the same idea that was used in Theorem 18 (handling several excerpts at once presents no new problems), but an exhaustive search for such "bonus" combinations, in arbitrarily many variables, would be less trivial.

CHAPTER 3

1. A "non-cancellation" lemma

In this Chapter we apply the results of the preceding chapters to give a lower bound for the length of the two-variable word

$$W(x, y) = [x, y]\left[[x^2, y^2], [x, y]\right]^N$$

where N is a positive integer. Here, to justify our notation, we stipulate that the word is to be taken as an element of M . We make the convention that $[x, y] = \bar{x}\bar{y}xy$ in M , but $[x, y] = \bar{x}\cdot\bar{y}\cdot x\cdot y$ in F , and furthermore we agree that, in F , an expression "$[x, y][u, v]$" means $[x, y]\cdot[u, v]$.

Our first step is to show that in any expression for a group-product of variables $x, y, \bar{x}, \bar{y}$ taking values in F , most of the cancellation-points can be removed immediately, and only certain combinations of non-trivial cancellations are possible. This will then be used to break our main problem into sub-cases.

THEOREM 28. *For* $x \neq 1$ *and* $y \neq 1$ *the possible contexts in which cancellation can take place between* $x, y, \bar{x}$ *and* $\bar{y}$ *are the following:*

$y\cdot x$	(A)
$\bar{y}\cdot x$	(B)
$x\cdot y$	(C)
$x\cdot\bar{y}$	(D)
$x\cdot x$	(E)
$y\cdot y$	(F)

Non-trivial cancellations in these contexts can take place only in the following combinations:

Case 0: *cancellation in none of these contexts*

Case 1: *cancellation only in* (E)

Case 2: *cancellation only in* (F)

Case 3: *cancellation only in* (E) *and* (F)

Case 4: *cancellation only in* (A)

Case 5: *cancellation only in* (B)

Case 6: *cancellation only in* (C)

Case 7: *cancellation only in* (D)

Case 8: *cancellation only in* (A) *and* (C)

Case 9: *cancellation only in* (B) *and* (D)

Case 10: *cancellation only in* (A), (D), *and* (E)

Case 11: *cancellation only in* (B), (C), *and* (E)

Case 12: *cancellation only in* (A), (B), *and* (F)

Case 13: *cancellation only in* (C), (D), *and* (F)

Case 14: *cancellation in all contexts* (A), (B), (C), (D), (E), *and* (F).

Proof. The first assertion is verified by inspection of the following table:

	x	$\bar{x}$	y	$\bar{y}$
x	E		C	D
$\bar{x}$		E	B	A
y	A	D	F	
$\bar{y}$	B	C		F

Figure 3

Evidently all the possible contexts are accounted for. To verify the second assertion, we consider first the case in which at most two mutually independent letters appear among the first and last letters of x and y. Let a be (denote) the first letter of x. If a letter independent of a occurs among the first and last letters of x and y, we may denote it by b.

Without loss of generality, we may suppose that the words x, y are of length 2 or greater. For, if $x = a$ then we may replace x by x^2 without affecting the contexts (A) to (F) in which non-trivial cancellations can or cannot take place; similarly if $\lambda(y) = 1$ we may replace y by y^2.

Again without loss of generality, we may assume that among the first and last letters of x and the first and last letters of y, taken in this order, the letter b occurs before the letter $\bar{b}$, unless neither b nor $\bar{b}$ occur. (If $\bar{b}$ occurs before b, relabel $b \to \bar{b}$.)

Figure 4

Ser No.	Trans.	Ser No.	Trans.	
1 2 3	$axa\ aya$ $\bar{a}$ b	9 28 29	$axa\ byb$ $axb\ \bar{a}y\bar{b}$ bya	Case 0: no cancellation
4 5 6	$\bar{a}ya$ $\bar{a}$ b	16 17 19	$ax\bar{a}\ \bar{a}yb$ bya byb	Case 1: cancellation in (E)
7 8 9	bya $\bar{a}$ b	10 25 32	$axa\ by\bar{b}$ $axb\ \bar{a}ya$ $by\bar{b}$	Case 2: cancellation in (F)
10 11	$\bar{b}$ $ax\bar{a}\ aya$	14 20	$ax\bar{a}\ \bar{a}ya$ $by\bar{b}$	Case 3: cancellations in (E), (F)
12 13 14	$\bar{a}$ b $\bar{a}ya$	8 26 30	$axa\ by\bar{a}$ $axb\ \bar{a}y\bar{a}$ $by\bar{a}$	Case 4: cancellation in (A)
15 16 17	$\bar{a}$ b bya	3 21 24	$axa\ ayb$ $axb\ aya$ $ay\bar{b}$	Case 5: cancellation in (B)
18 19 20	$\bar{a}$ b $\bar{b}$	6 33 36	$axa\ \bar{a}yb$ $axb\ \bar{b}ya$ $\bar{b}y\bar{b}$	Case 6: cancellation in (C)
21 22 23	$axb\ aya$ $\bar{a}$ b	7 27 31	$axa\ bya$ $axb\ \bar{a}yb$ byb	Case 7: cancellation in (D)
24 25	$\bar{b}$ $\bar{a}ya$	5 34	$axa\ \bar{a}y\bar{a}$ $axb\ \bar{b}y\bar{a}$	Case 8: cancellation in (A), (C)
26 27	$\bar{a}$ b	1 23	$axa\ aya$ $axb\ ayb$	Case 9: cancellation in (B), (D)
28 29	$\bar{b}$ bya	15 18	$ax\bar{a}\ \bar{a}y\bar{a}$ $by\bar{a}$	Case 10: cancellation in (A), (D), (E)
30 31	$\bar{a}$ b	11 13	$ax\bar{a}\ aya$ ayb	Case 11: cancellation in (B), (C), (E)
32 33	$\bar{b}$ $\bar{b}ya$	2 22	$axa\ ay\bar{a}$ $axb\ ay\bar{a}$	Case 12: cancellation in (A), (B), (F)
34 35	$\bar{a}$ b	4 35	$axa\ \bar{a}ya$ $axb\ \bar{b}yb$	Case 13: cancellation in (C), (D), (F)
36	$\bar{b}$	12	$ax\bar{a}\ ay\bar{a}$	Case 14: cancellation in (A), (B), (C), (D), (E), (F)

The table given in Figure 4 lists the 36 possible transcriptions of (x, y) making these letters explicit, subject to the above assumptions. The transcriptions (or rather, their results) are listed first in "lexicographical" order (and numbered from 1 to 36), and then grouped according to the cases (0 to 14) specifying in which of the contexts (A)-(F) non-trivial cancellation takes place. In listing the transcriptions, we have omitted from each line information that is unchanged from the previous line.

Thus, the first transcription listed is $(x, y) \to (axa, aya)$, and it falls under Case 9: non-trivial cancellation takes place only in the contexts (B) and (D). The second transcription is $(x, y) \to (axa, ay\bar{a})$, and it falls under Case 12: non-trivial cancellation takes place only in the contexts (A), (B), and (F), and so on.

Inspection of the table shows that all the transcriptions (subject to our assumptions) are accounted for, and that cases 0 to 14 exactly describe the combinations of contexts (A)-(F) in which non-trivial cancellations take place.

If three or more mutually independent letters occur among the first and last letters of the words x and y , we may without loss of generality consider as representative the four transcriptions

(37): $(x, y) \to (axb, cy)$

(38): $(x, y) \to (axb, yc)$

(39): $(x, y) \to (ax, byc)$

(40): $(x, y) \to (xa, byc)$

where a, b, c are pairwise independent letters. (The numbering follows on from the last transcription listed in Figure 4.)

After the transcription (37), non-trivial cancellation is possible only in the contexts (A): $cy \cdot axb$, (D): $axb \cdot \bar{y}\bar{c}$, and (F): $cy \cdot cy$; moreover, since a, b, c are pairwise independent, a non-trivial cancellation in one of these contexts rules out non-trivial cancellation in the other two. Therefore the possible non-trivial cancellations fall under Case 4, Case 7, or Case 2 of our list. Similarly, after the transcription (38) non-trivial cancellation is possible in at most one of the contexts (B): $\bar{c}\bar{y} \cdot axb$, (C): $axb \cdot yc$, or (F): $yc \cdot yc$; after the transcription (39) non-trivial cancellation is possible in at most one of (C): $ax \cdot byc$, (D): $ax \cdot \bar{c}\bar{y}\bar{b}$, or

(E): $ax \cdot ax$; and, finally, after the transcription (40) non-trivial cancellation is possible in at most one of (A): $byc \cdot xa$, (B): $\bar{c}\bar{y}\bar{b} \cdot xa$, or (E): $xa \cdot xa$. Thus, if three or more mutually independent letters occur among the first and last letters of x and y , the possible non-trivial cancellations among $x, \bar{x}, y$ and $\bar{y}$ fall under one of Case 1, Case 2, Case 4, Case 5, Case 6, Case 7. This completes the proof. //

COROLLARY 28.1. *If* $x \neq 1$ *and* $y \neq 1$ *and if non-trivial cancellation takes place in each of the contexts* $y \cdot x, \bar{y} \cdot x, x \cdot y, x \cdot \bar{y}$, $x \cdot x$ *and* $y \cdot y$ *, then there are words* $u \neq 1$, $v \neq 1$, $z \neq 1$ *such that* $x = zu\bar{z}$, $y = zv\bar{z}$ *, and non-trivial cancellation is ruled out in at least one of the contexts* $v \cdot u$, $\bar{v} \cdot u$, $u \cdot v$, $u \cdot \bar{v}$, $u \cdot u$, $v \cdot v$.

Proof. Inspection of Figure 4 (Case 14) shows that if x and y satisfy the above hypotheses then $x = a_1x_1\bar{a}_1$, $y = a_1y_1\bar{a}_1$ for some words x_1, y_1 and some letter a_1 . If x_1, y_1 do not satisfy the hypotheses (with x, y replaced by x_1, y_1) then $z = a_1$, $u = x_1$, $v = y_1$ are the words we seek; otherwise we have $x_1 = a_2x_2\bar{a}_2$, $y_1 = a_2y_2\bar{a}_2$ for some letter a_2 , and so forth. By Theorem 6, each of the words $x_1, x_2, \ldots$ and each of the words $y_1, y_2, \ldots$ differs from 1 . Since $\lambda(x), \lambda(y)$ are finite, we reach in finitely many steps a pair of words (x_n , y_n) such that non-trivial cancellation is ruled out in at least one of the contexts $y_n \cdot x_n , \bar{y}_n \cdot x_n , x_n \cdot y_n$, $x_n \cdot \bar{y}_n , x_n \cdot x_n , y_n \cdot y_n$. But now the words $z = a_1 \ldots a_n$, $u = x_n$, $v = y_n$ satisfy our requirements. //

COROLLARY 28.2. *If* $x \neq 1$ *and* $y \neq 1$ *then, in studying possible cancellation (rather than actual non-trivial cancellation) between* $x, y, \bar{x}, \bar{y}$ *, it is sufficient to consider* Case 3 *and* Cases 8 *to* 14 *in the list of cases enunciated in the above theorem.*

Proof. The question that concerns us is, in which of the contexts (A)-(F) can the cancellation-points be removed? Setting aside Case 14, in which non-trivial cancellation actually takes place in all of these contexts, we see that Case 1 and Case 2 may be subsumed under Case 3, Case 4 and Case 6 may be subsumed under Case 8, and, finally, Case 5 and Case 7 may be subsumed under Case 9. Of course, Case 0 is subsumed under all of the other cases. //

2. The main theorem

THEOREM 29. *Suppose that* x, y *are arbitrary words in* F , *and that* N *is an arbitrary positive integer. Let* $W(x, y)$ *be defined by*

$$W(x, y) = [x, y]\big[[x{\cdot}x,\ y{\cdot}y],\ [x, y]\big]^N .$$

If $x{\cdot}y = y{\cdot}x$ *then* $W(x, y) = 1$. *Otherwise, there is a word* z *in* F , *obtainable algorithmically from* x *and* y , *such that*

$$\lambda\big(W(x, y)^*\big) \geq 4 + 8N + 4N\big(\lambda(x^z) + (y^z)\big) .$$

Proof. The expression $W(x, y)$, multiplied out, is

$$\bar{x}{\cdot}\bar{y}{\cdot}\{{\cdot}x{\cdot}\bar{y}{\cdot}\bar{x}{\cdot}\bar{x}{\cdot}y{\cdot}y{\cdot}x{\cdot}x{\cdot}\bar{y}{\cdot}\bar{x}{\cdot}y{\cdot}\bar{x}{\cdot}\bar{y}{\cdot}\bar{y}{\cdot}x{\cdot}x{\cdot}y{\cdot}y{\cdot}\bar{x}{\cdot}\bar{y}{\cdot}\}^N{\cdot}x{\cdot}y .$$

The "redundant" cancellation-points shown here indicate separately that cancellation is possible

(i) between the word $\bar{x}{\cdot}\bar{y}$ appearing outside the braces, and the rest of the expression,

(ii) between two consecutive repetitions of that part of the expression appearing in braces,

(iii) between the word $x{\cdot}y$ appearing outside the braces, and the rest of the expression.

(A fine point of notation - which we shall not require in the present work - suggests itself here:

$$u{\cdot}({\cdot}v{\cdot})^2w = u{\cdot}v{\cdot}vw ,$$

but

$$u{\cdot}({\cdot}v{\cdot})^2{\cdot}w = u{\cdot}v{\cdot}v{\cdot}w ,$$

and so forth.)

We begin by re-labelling the variables as x_0, y_0 instead of x, y respectively, and we present $W(x_0, y_0)^*$ in the form corresponding to $y_0{\cdot}W(x_0, y_0){\cdot}\bar{y}_0$, thus:

$$W(x_0, y_0)^* = {\cdot}y_0{\cdot}\bar{x}_0{\cdot}\{{\cdot}\bar{y}_0{\cdot}x_0{\cdot}\bar{y}_0{\cdot}\bar{x}_0{\cdot}\bar{x}_0{\cdot}y_0{\cdot}y_0{\cdot}x_0{\cdot}x_0{\cdot}\bar{y}_0{\cdot}$$
$${\cdot}\bar{x}_0{\cdot}y_0{\cdot}\bar{x}_0{\cdot}\bar{y}_0{\cdot}\bar{y}_0{\cdot}x_0{\cdot}x_0{\cdot}y_0{\cdot}\bar{x}_0{\cdot}\}^N{\cdot}\bar{y}_0{\cdot}x_0{\cdot}{}^* ,$$

(the cancellation-points fore and aft of this expression have an obvious interpretation). Henceforth we write W for $W(x_0, y_0)$, W^* for $W(x_0, y_0)^*$.

Suppose first that x_0, y_0 are coradical. Then clearly x_0, y_0

are powers (in the sense of group multiplication in F) of some common root. Therefore they commute in F , and we have $W = 1$.

Suppose now that x_0, y_0 are not coradical. We shall consider the possible cancellations in the above expression for W^* , under the cases specified by Corollary 28.2. For ease of reference, we use the same labels (A)-(F) to identify contexts in which cancellation can take place, and we preserve the numbering of cases as established in Theorem 28 and Corollary 28.2 - thus we shall begin with "Case 3", and then proceed with "Case 8". When necessary, we divide these cases into "sub-cases", which will be indicated by a decimal-fraction numbering.

Case 3. Non-trivial cancellation is restricted to one or possibly both of the contexts (E): $x_0 \cdot x_0$ and (F): $y_0 \cdot y_0$. In this case we have

$$W^* = y_0\bar{x}_0\{\bar{y}_0x_0\bar{y}_0(\bar{x}_0\cdot\bar{x}_0)(y_0\cdot y_0)(x_0\cdot x_0)\bar{y}_0\bar{x}_0y_0\bar{x}_0(\bar{y}_0\cdot\bar{y}_0)$$
$$(x_0\cdot x_0)(y_0\cdot y_0)\bar{x}_0\}^{N-}\bar{y}_0x_0{}^*$$

and since $x_0 \neq 1$, $y_0 \neq 1$ we have, by Corollary 11.1,

$$\begin{aligned}\lambda(W^*) &\geq 2\{\lambda(x_0)+\lambda(y_0)\} + N\{7\lambda(x_0)+7\lambda(y_0)+6\} \\ &> 4 + 8N + 4N\{\lambda(x_0)+\lambda(y_0)\} .\end{aligned}$$

Thus, our assertion holds with $z = 1$.

Case 8. Non-trivial cancellation is restricted to (one or both of) the two contexts (A): $y_0 \cdot x_0$ and (C): $x_0 \cdot y_0$. We have

$$W^* = \cdot y_0\bar{x}_0\cdot\{\cdot\bar{y}_0x_0\bar{y}_0\cdot\bar{x}_0{}^2y_0{}^2\cdot x_0{}^2\bar{y}_0\cdot\bar{x}_0y_0\bar{x}_0\cdot\bar{y}_0{}^2x_0{}^2\cdot y_0{}^2\bar{x}_0\cdot\}^N\cdot\bar{y}_0x_0\cdot{}^*$$

and now we consider sub-cases according to whether cancellation dies in $y_0 \cdot x_0$, $x_0 \cdot y_0$, or neither.

Case 8.1. If cancellation dies in $y_0 \cdot x_0$, we transcribe $(x_0, y_0) \to (zx, y\bar{z})$, where $x \neq 1$, $y \neq 1$ and $y \cdot x = yx$. Now we have

$$W^* = \cdot y\bar{z}\bar{x})\{\bar{y}z\{xz\bar{y}\cdot(\bar{x}\bar{z})^2y\}\bar{z}yxz(xz\bar{y}\cdot\bar{x}\bar{z}y)\bar{z}\bar{x}\bar{y}z\{\bar{y}(zx)^2\cdot(y\bar{z})^2\bar{x}\}\}^N(\bar{y}zx\cdot{}^*$$

where the "reversed" brackets outside the braces serve to emphasize that the word $\cdot y\bar{z}\bar{x}$ with which the above expression begins is actually adjacent in the circular expression to the word $\bar{y}zx$ that is written at the end. By Theorem 20, since zx and $y\bar{z}$ are not coradical, we have

$$\begin{aligned}\lambda(W^*) &\geq 4 + N\{3\lambda(\bar{z}y)+2\lambda(xz)+\lambda(zx)+4+4+\lambda(zx)+\lambda(y\bar{z})+4\} \\ &= 4 + 12N + 4N\{\lambda(x_0)+\lambda(y_0)\} .\end{aligned}$$

Case 8.2. If cancellation dies in $x_0 \cdot y_0$, we transcribe $(x_0, y_0) \to (xz, \bar{z}y)$, where $x \neq 1$, $y \neq 1$ and $x \cdot y = xy$. Now we have

$$W^* = (y\bar{z}\bar{x} \cdot \{ \cdot \bar{y}zx)z\bar{y}\bar{x}\bar{z}\big(\bar{x}(\bar{z}y)^2 \cdot (xz)^2\bar{y}\big)\bar{x}\bar{z}\big(y\bar{z}\bar{x} \cdot (\bar{y}z)^2x\big)zxy\bar{z}(y\bar{z}\bar{x} \cdot \}^N \cdot \bar{y}zx)^*$$

where the "interlacing" of the braces and the brackets is to be interpreted as bracketing together the word $y\bar{z}\bar{x}$ with which the whole expression begins with the word $\bar{y}zx$ immediately following (at the beginning of the first occurrence of the portion between braces, which is repeated N times), also bracketing together the word $y\bar{z}\bar{x}$ at the end of the N times repeated portion and the word $\bar{y}zx$ at the beginning of (the next occurrence of) this repeated portion, and finally bracketing together the word $y\bar{z}\bar{x}$ at the end of the last repetition of the portion in braces and the word $\bar{y}zx$ outside the braces, at the end of the whole expression. Since xz and $\bar{z}y$ are not coradical, we may again apply Theorem 20, to find

$$\begin{aligned} \lambda(W^*) &\geq 4 + N\{3\lambda(zx)+2\lambda(y\bar{z})+\lambda(\bar{z}y)+\lambda(xz)+4+\lambda(\bar{z}y)+4+4\} \\ &= 4 + 12N + 4N\big(\lambda(x_0)+\lambda(y_0)\big) . \end{aligned}$$

The reader will note that there are two ways of making this computation. One may take into account separately the contribution of $y\bar{z}\bar{x} \cdot \bar{y}zx$ at the beginning of the whole expression (length at least 4), add $N - 1$ times the contribution of $y\bar{z}\bar{x} \cdot \bar{y}zx$ "between" the repetitions of the portion shown between braces (total length at least $4N - 4$), then add the contribution of $y\bar{z}\bar{x} \cdot \bar{y}zx$ at the end of the whole expression (length at least 4), and finally add N times the minimum contribution of whatever lies wholly within the braces. Alternatively, in the present case one may observe that the braces can be "moved" to one side, to give for example

$$W^* = (y\bar{z}\bar{x} \cdot \bar{y}zx)\{z\bar{y}\bar{x} \ \ldots \ xy\bar{z}(y\bar{z}\bar{x} \cdot \bar{y}zx)\}^{N*}$$

where there is no "interlacing" of brackets and braces as described above. Of course, the trick of "moving" the braces is available to us only when the symbols outside the right-hand brace are the same as those immediately following the left-hand brace or when the symbols outside the left-hand brace are the same as those immediately preceding the right-hand brace.

Case 8.3. If cancellation persists in $y_0 \cdot x_0$ and in $x_0 \cdot y_0$, by Theorem 9 we may transcribe $(x_0, y_0) \to \big((xy)^m x, (\bar{x}\bar{y})^n \bar{x}\big)$, where x, y are not coradical and m, n are non-negative integers differing by 1 . We consider separately the cases $m = n + 1$ (with $n = 0$ or

$n > 0$) and $n = m + 1$ (with $m = 0$ or $n > 0$).

Case 8.31. If $m = n + 1$ we have, after cancellation,

$$W^* = \cdot\bar{x}\bar{y}\cdot\{\cdot(xy)^m x\cdot\bar{y}\bar{x}(\bar{x}\bar{y})^m\bar{x}(\bar{x}\bar{y})^n\bar{x}\cdot yx(xy)^m x\cdot\bar{y}\bar{x}(\bar{x}\bar{y})^n\bar{x}^2\bar{y}\cdot(xy)^n x(xy)^m x^2 y\cdot$$
$$\cdot(\bar{x}\bar{y})^n\bar{x}^2\bar{y}\cdot\}^N\cdot xy\cdot * \ .$$

We consider separately the cases $n > 0$, $n = 0$.

Case 8.311. If $n > 0$ we have

$$W^* = \cdot\bar{x}\bar{y}\cdot\{\cdot xy)(xy)^{n-1}x(yx\cdot\bar{y}\bar{x})(\bar{x}\bar{y})^m\bar{x}$$
$$(\bar{x}\bar{y})^{n-1}\bar{x}(\bar{y}\bar{x}\cdot yx)(xy)^n x(yx\cdot\bar{y}\bar{x})(\bar{x}\bar{y})^n\bar{x}(\bar{x}\bar{y}\cdot xy)$$
$$(xy)^{n-1}x(xy)^m x(xy\cdot\bar{x}\bar{y})(\bar{x}\bar{y})^{n-1}\bar{x}(\bar{x}\bar{y}\cdot\}^N\cdot xy\cdot * \ .$$

By Theorem 12 and Theorem 21, cancellation dies in the bracketed portions. Observing that $\lambda(x_0) + \lambda(y_0) = 2\lambda\{(xy)^n x\} + \lambda(xy)$, and using the estimates given by Theorem 21 and Theorem 22, we find

$$\lambda(W^*) = \lambda(\bar{x}\bar{y}\cdot xy\cdot\bar{x}\bar{y}\cdot xy) - \lambda(\bar{x}\bar{y}\cdot xy) + N\{2\lambda\{(xy)^{n-1}x\}+6\lambda\{(xy)^n x\}$$
$$+3\lambda(xy\cdot\bar{x}\bar{y})+3\lambda(\bar{x}\bar{y}\cdot xy)\}$$
$$\geq 4 + N\{8\lambda\{(xy)^n x\}-2\lambda(xy)+3\{2\lambda(xy)+4\}\}$$
$$= 4 + 12N + 4N\{\lambda(x_0)+\lambda(y_0)\} \ .$$

Case 8.312. If $n = 0$, $m = 1$ we have

$$W^* = \cdot\bar{x}\bar{y}\cdot\{\cdot xyx\cdot\bar{y}\bar{x}^2\bar{y}\bar{x}^2\cdot yx^2yx\cdot\bar{y}\bar{x}^3\bar{y}\cdot x^2yx^2y\cdot\bar{x}^2\bar{y}\cdot\}^N\cdot xy\cdot *$$

with $x_0 = xyx$, $y_0 = \bar{x}$ and x, y not coradical. That is, $x_0 = \bar{y}_0 y\bar{y}_0$. Now we transcribe $y \to x^{k-1}yx^{k-1}$ $(= \bar{y}_0{}^{k-1}y\bar{y}_0{}^{k-1})$, where the positive integer k is chosen as large as possible. (If $k = 1$, this transcription changes nothing.) We now have

$$W^* = \cdot\bar{x}\bar{y}\cdot\{\cdot xyx\cdot\bar{y}\bar{x})\bar{x}^{2k-1}(\bar{y}\bar{x}^2\cdot yx)x^{2k-1}$$
$$(yx\cdot\bar{y}\bar{x})\bar{x}^{2k-1}(\bar{x}\bar{y}\cdot x^2y)x^{2k-1}(xy\cdot\bar{x}^2\bar{y}\cdot\}^N\cdot xy\cdot *$$

where $x_0 = x^k yx^k$, $y_0 = \bar{x}$, and clearly x, y are again not coradical (else so are x_0, y_0). We consider separately the cases in which cancellation dies in $\bar{y}\cdot x$, $x\cdot\bar{y}$, or neither.

Case 8.3121. If cancellation dies in $\bar{y}\cdot x$, we may transcribe $(x, y) \to (zx, zy)$, where after the transcription we have $\bar{y}\cdot x = \bar{y}x$. Now we have

$$W^* = \cdot\bar{x}\bar{z}\bar{y})\{xz(yzx\cdot\bar{y}\bar{z}\bar{x})(\bar{z}\bar{x})^{2k-1}\bar{z}\bar{y}(\bar{z}\bar{x})^2$$
$$yz(xz)^{2k}(yzx\cdot\bar{y}\bar{z}\bar{x})(\bar{z}\bar{x})^{2k}\bar{z}\bar{y}(xz)^2yz(xz)^{2k-1}\{xzy\cdot(\bar{x}\bar{z})^2\bar{y}\}\}^N(xzy\cdot *$$

where $x_0 = (zx)^k zy(zx)^k$, $y_0 = \bar{x}\bar{z}$, and zx, zy are not coradical. By Theorem 20, we have

$$\begin{aligned}\lambda(W^*) &\geq 4 + N\{4+\lambda(x_0)+2\lambda(y_0)+\lambda(x_0)+4+\lambda(x_0)+\lambda(y_0)+\lambda(x_0)+\lambda(y_0)+4\} \\ &= 4 + 12N + 4N\big(\lambda(x_0)+\lambda(y_0)\big) .\end{aligned}$$

Case 8.3122. If cancellation dies in $x\cdot\bar{y}$, we may transcribe $(x, y) \to (xz, yz)$, where after the transcription we have $x\cdot\bar{y} = x\bar{y}$. Now we have

$$\begin{aligned}W^* = (\bar{x}\bar{z}\bar{y}\cdot\{\cdot xzy)zx\bar{y}\bar{z}(\bar{x}\bar{z})^{2k}\big(\bar{y}(\bar{z}\bar{x})^2\cdot yzx\big) \\ (zx)^{2k-1}zyzx\bar{y}\bar{z}(\bar{x}\bar{z})^{2k}(\bar{x}\bar{z}\bar{y}\cdot(xz)^2y)(zx)^{2k}zy\bar{x}\bar{z}(\bar{x}\bar{z}\bar{y}\cdot\}^N\cdot xzy)^*\end{aligned}$$

where $x_0 = (xz)^k yz(xz)^k$, $y_0 = \bar{z}\bar{x}$, and xz, yz are not coradical. By Theorem 20, we have

$$\begin{aligned}\lambda(W^*) &\geq 4 + N\{\lambda(y_0)+\lambda(x_0)+\lambda(y_0)+4+2\lambda(x_0)+\lambda(y_0)+4+\lambda(x_0)+\lambda(y_0)+4\} \\ &= 4 + 12N + 4N\big(\lambda(x_0)+\lambda(y_0)\big) .\end{aligned}$$

Case 8.3123. If cancellation persists in $\bar{y}\cdot x$ and in $x\cdot\bar{y}$, by Theorem 9 we may transcribe $(x, y) \to \big((xy)^r x, (xy)^s x\big)$, where r, s are non-negative integers differing by 1 . After this transcription x, y are not coradical, and we have

$$x_0 = \big((xy)^r x\big)^k (xy)^s x\big((xy)^r x\big)^k \quad , \quad y_0 = (\bar{x}\bar{y})^r \bar{x} .$$

We consider separately the cases $r = s + 1$ and $s = r + 1$.

Case 8.31231. If $r = s + 1$ we have, after cancellation,

$$\begin{aligned}W^* = \cdot\bar{y}\bar{x}\cdot\{yx)(xy)^s x(xy\cdot\bar{x}\bar{y})(\bar{x}\bar{y})^s\bar{x}\big((\bar{x}\bar{y})^r\bar{x}\big\}^{2k-1} \\ (\bar{x}\bar{y})^s\bar{x}(\bar{x}\bar{y})^r\bar{x}(\bar{x}\bar{y}\cdot xy)(xy)^s x\big((xy)^r x\big\}^{2k-1}(xy)^s x \\ (xy\cdot\bar{x}\bar{y})(\bar{x}\bar{y})^s\bar{x}\big((\bar{x}\bar{y})^r\bar{x}\big\}^{2k-1}(\bar{x}\bar{y})^s\bar{x}(\bar{y}\bar{x}\cdot yx)(xy)^r x(xy)^s x \\ \big((xy)^r x\big\}^{2k-1}(xy)^s x(yx\cdot\bar{y}\bar{x})(\bar{x}\bar{y})^s\bar{x}(\bar{y}\bar{x}\cdot\}^N\cdot yx\cdot{}^* .\end{aligned}$$

By Theorems 12 and 21 cancellation dies in the bracketed portions. With Theorem 22, and the observation that $\lambda(x_0) + \lambda(y_0) = 2(k+1)\lambda\big((xy)^r x\big) - \lambda(xy)$, we now find

$$\begin{aligned}\lambda(W^*) &= \lambda(\bar{y}\bar{x}\cdot yx\cdot\bar{y}\bar{x}\cdot yx) - \lambda(\bar{y}\bar{x}\cdot yx) \\ &\qquad + N\big\{10\lambda\big((xy)^s x\big)+(8k-2)\lambda\big((xy)^r x\big)+3\lambda(xy\cdot\bar{x}\bar{y})+3\lambda(\bar{x}\bar{y}\cdot xy)\big\} \\ &\geq 4 + N\big\{8(k+1)\lambda\big((xy)^r x\big)-10\lambda(xy)+3\big(2\lambda(xy)+4\big)\big\} \\ &= 4 + 12N + 4N\big(\lambda(x_0)+ (y_0)\big) .\end{aligned}$$

Case 8.31232. If $s = r + 1$ we have, after cancellation,

$$\begin{aligned}W^* = \cdot\bar{x}\bar{y}\cdot\{\cdot xy)(xy)^{r-1}x(yx\cdot\bar{y}\bar{x})\big((\bar{x}\bar{y})^r\bar{x}\big\}^{2k} \\ (\bar{x}\bar{y})^s\bar{x}(\bar{x}\bar{y})^{r-1}\bar{x}(\bar{y}\bar{x}\cdot yx)\big((xy)^r x\big\}^{2k+1}(yx\cdot\bar{y}\bar{x}) \\ \big((\bar{x}\bar{y})^r\bar{x}\big\}^{2k+1}(\bar{x}\bar{y}\cdot xy)(xy)^{r-1}x(xy)^r x\big((xy)^r x\big\}^{2k} \\ (xy\cdot\bar{x}\bar{y})(\bar{x}\bar{y})^{r-1}\bar{x}(\bar{x}\bar{y}\cdot\}^N\cdot xy\cdot{}^* .\end{aligned}$$

Note that the case $r = 0$, $s = 1$ does not arise, as this would imply

$x_0 = x^{k+1}yx^{k+1}$, $y_0 = \bar{x}$, contrary to the definition of k as the largest integer such that $x_0 = \bar{y}_0{}^k y \bar{y}_0{}^k$ for some y . Thus $r - 1$ is non-negative, and the above expression makes sense.

Now cancellation dies in the bracketed portions, so, applying Theorems 21 and 22 and observing that $\lambda(x_0) + \lambda(y_0) = 2(k+1)\lambda\big((xy)^r x\big) + \lambda(xy)$, we find

$$\begin{aligned}
\lambda(W^*) &= \lambda(\bar{x}\bar{y}\cdot xy\cdot \bar{x}\bar{y}\cdot xy) - \lambda(\bar{x}\bar{y}\cdot xy) + N\big\{2\lambda\big((xy)^{r-1}x\big)+(8k+6)\lambda\big((xy)^r x\big) \\
&\qquad\qquad +3\lambda(\bar{x}\bar{y}\cdot xy)+3\lambda(xy\cdot \bar{x}\bar{y})\big\} \\
&\geq 4 + N\big\{8(k+1)\lambda\big((xy)^r x\big)-2\lambda(xy)+3\big(2\lambda(xy)+4\big)\big\} \\
&= 4 + 12N + 4N\big(\lambda(x_0)+\lambda(y_0)\big) .
\end{aligned}$$

This completes our discussion of Case 8.312, and hence also of Case 8.31. We proceed with

Case 8.32: in which $n = m + 1$. We have, after cancellation,

$$\begin{aligned}
W^* = \cdot\bar{y}\bar{x}\cdot\{&\cdot yx(xy)^m x^2 y\cdot(\bar{x}\bar{y})^m \bar{x}^2 (\bar{y}\bar{x})^n \bar{x}\bar{y}\cdot(xy)^m x^2 y\cdot(\bar{x}\bar{y})^n \bar{x}\cdot yx(xy)^n x^2 (yx)^m\cdot \\
&\cdot\bar{y}\bar{x}(\bar{x}\bar{y})^n \bar{x}\cdot\}^N\cdot yx\cdot * .
\end{aligned}$$

We consider separately the cases $m > 0$ and $m = 0$.

Case 8.321. If $m > 0$, we have

$$\begin{aligned}
W^* = \cdot\bar{y}\bar{x}\cdot\{&\cdot yx)(xy)^m x(xy\cdot\bar{x}\bar{y})(\bar{x}\bar{y})^{m-1}\bar{x}(\bar{x}\bar{y})^n \bar{x} \\
&(\bar{x}\bar{y}\cdot xy)(xy)^{m-1}x(xy\cdot\bar{x}\bar{y})(\bar{x}\bar{y})^{m-1}\bar{x}(\bar{y}\bar{x}\cdot yx)(xy)^n x \\
&(xy)^{m-1}x(yx\cdot\bar{y}\bar{x})(\bar{x}\bar{y})^m \bar{x}(\bar{y}\bar{x}\cdot\}^N\cdot yx\cdot * .
\end{aligned}$$

Cancellation dies in the bracketed portions, and we have

$$\begin{aligned}
\lambda(W^*) &= \lambda(\bar{y}\bar{x}\cdot yx\cdot\bar{y}\bar{x}\cdot yx) - \lambda(\bar{y}\bar{x}\cdot yx) + N\big\{2\lambda\big((xy)^{m-1}x\big)+6\lambda\big((xy)^m x\big) \\
&\qquad\qquad +3\lambda(xy\cdot\bar{x}\bar{y})+3\lambda(\bar{x}\bar{y}\cdot xy)\big\} \\
&\geq 4 + N\big\{8\lambda\big((xy)^m x\big)-2\lambda(xy)+3\big(2\lambda(xy)+4\big)\big\} \\
&= 4 + 12N + 4N\big(\lambda(x_0)+\lambda(y_0)\big) ,
\end{aligned}$$

by Theorems 21, 22.

Case 8.322. If $m = 0$, $n = 1$ we have

$$W^* = \cdot\bar{y}\bar{x}\cdot\{\cdot yx^3 y\cdot\bar{x}^2\bar{y}\bar{x}^2\bar{y}\cdot x^2 y\cdot\bar{x}\bar{y}\bar{x}\cdot yx^2 yx^2\cdot\bar{y}\bar{x}^2\bar{y}\bar{x}\cdot\}^N\cdot yx\cdot *$$

with $x_0 = x$, $y_0 = \bar{x}\bar{y}\bar{x}$ and x, y not coradical. That is, $y_0 = \bar{x}_0\bar{y}\bar{x}_0$. We now transcribe $y \to x^{k-1}yx^{k-1}$, choosing the positive integer k as large as possible. Now we have

$$W^* = \cdot\bar{y}\bar{x}\cdot\{\cdot yx^{2k+1}y\cdot\bar{x}^2\bar{y}\bar{x}^{2k}\bar{y}\cdot x^2 y\cdot\bar{x}\bar{y}\bar{x}\cdot yx^{2k}yx^2\cdot\bar{y}\bar{x}^{2k}\bar{y}\bar{x}\cdot\}^N\cdot yx\cdot *$$

where $x_0 = x$, $y_0 = \bar{x}^k\bar{y}\bar{x}^k$, and again x, y are not coradical. We consider separately the cases in which cancellation dies in $\bar{y}\cdot x$, $x\cdot\bar{y}$, or neither.

Case 8.3221. If cancellation dies in $\bar{y}\cdot x$, we may transcribe $(x, y) \to (zx, zy)$, where after the transcription we have $\bar{y}\cdot x = \bar{y}x$. Now we have

$$W^* = \cdot\bar{y}\bar{z}\bar{x})\{yz(xz)^{2k}\left[xzy\cdot(\bar{x}\bar{z})^2\bar{y}\right](\bar{z}\bar{x})^{2k}\bar{z}\bar{y}$$
$$xz(xzy\cdot\bar{x}\bar{z}\bar{y})\bar{z}\bar{x}yz(xz)^{2k}\left[y(zx)^2\cdot\bar{y}\bar{z}\bar{x}\right](\bar{z}\bar{x})^{2k-1}\bar{z}\bar{y}\bar{z}\bar{x}\}^N(yzx\cdot *$$

where $x_0 = zx$, $y_0 = (\bar{x}\bar{z})^k\bar{y}\bar{z}(\bar{x}\bar{z})^k$, and zx, zy are not coradical. By Theorem 20 we have

$$\lambda(W^*) \geq 4 + N\{\lambda(y_0)+\lambda(x_0)+4+\lambda(y_0)+\lambda(x_0)+4+\lambda(x_0)+\lambda(y_0)+\lambda(x_0)+4+\lambda(y_0)\}$$
$$= 4 + 12N + 4N\left(\lambda(x_0)+\lambda(y_0)\right) .$$

Case 8.3222. If cancellation dies in $x\cdot\bar{y}$, we may transcribe $(x, y) \to (xz, yz)$, where after the transcription we have $x\cdot\bar{y} = x\bar{y}$. Now we have

$$W^* = (\bar{y}\bar{z}\bar{x}\cdot\{\cdot yzx)(zx)^{2k}zy(\bar{x}\bar{z})^2\bar{y}\bar{z}(\bar{x}\bar{z})^{2k-1}$$
$$\left[\bar{x}\bar{z}\bar{y}\cdot(xz)^2y\right]\bar{x}\bar{z}(\bar{y}\bar{z}\bar{x}\cdot yzx)(zx)^{2k-1}zy(zx)^2\bar{y}\bar{z}(\bar{x}\bar{z})^{2k}(\bar{y}\bar{z}\bar{x}\cdot\}^N\cdot yzx)^*$$

where $x_0 = xz$, $y_0 = (\bar{z}\bar{x})^k\bar{z}\bar{y}(\bar{z}\bar{x})^k$, and xz, yz are not coradical. By Theorem 20, we find as before

$$\lambda(W^*) \geq 4 + N\{\lambda(y_0)+\lambda(x_0)+\lambda(y_0)+\lambda(x_0)+4+\lambda(x_0)+4+\lambda(y_0)+\lambda(x_0)+\lambda(y_0)+4\}$$
$$= 4 + 12N + 4N\left(\lambda(x_0)+\lambda(y_0)\right) .$$

Case 8.3223. If cancellation persists in $\bar{y}\cdot x$ and in $x\cdot\bar{y}$, by Theorem 9 we may transcribe $(x, y) \to \left[(xy)^r x, (xy)^s x\right]$, where r, s are non-negative integers differing by 1 . After this transcription x, y are not coradical, and we have

$$x_0 = (xy)^r x , \quad y_0 = \left[(\bar{x}\bar{y})^r\bar{x}\right]^k(\bar{x}\bar{y})^s\bar{x}\left[(\bar{x}\bar{y})^r\bar{x}\right]^k .$$

We consider separately the cases $r = s + 1$ and $s = r + 1$.

Case 8.32231. If $r = s + 1$ we have, after cancellation,

$$W^* = \cdot\bar{x}\bar{y}\cdot\{\cdot xy)(xy)^s x\left[(xy)^r x\right]^{2k-1}(xy)^s x(yx\cdot\bar{y}\bar{x})$$
$$(\bar{x}\bar{y})^r\bar{x}(\bar{x}\bar{y})^s\bar{x}\left[(\bar{x}\bar{y})^r\bar{x}\right]^{2k-1}(\bar{x}\bar{y})^s\bar{x}(\bar{y}\bar{x}\cdot yx)(xy)^s x$$
$$(yx\cdot\bar{y}\bar{x})(\bar{x}\bar{y})^s\bar{x}(\bar{x}\bar{y}\cdot xy)(xy)^s x\left[(xy)^r x\right]^{2k-1}(xy)^s x(xy)^r x$$
$$(xy\cdot\bar{x}\bar{y})(\bar{x}\bar{y})^s\bar{x}\left[(\bar{x}\bar{y})^r\bar{x}\right]^{2k-1}(\bar{x}\bar{y})^s\bar{x}(\bar{x}\bar{y}\cdot\}^N\cdot xy\cdot * .$$

By Theorems 12, 21 and 22 we find that cancellation dies in the bracketed portions, and

$$\lambda(W^*) = \lambda(\bar{x}\bar{y}\cdot xy\cdot\bar{x}\bar{y}\cdot xy) - \lambda(\bar{x}\bar{y}\cdot xy) + N\{10\lambda\left[(xy)^s x\right]+(8k-2)\lambda\left[(xy)^r x\right]$$
$$+3\lambda(xy\cdot\bar{x}\bar{y})+3\lambda(\bar{x}\bar{y}\cdot xy)\}$$
$$\geq 4 + N\{8(k+1)\lambda\left[(xy)^r x\right]-10\lambda(xy)+3\left(2\lambda(xy)+4\right)\}$$
$$= 4 + 12N + 4N\left(\lambda(x_0)+\lambda(y_0)\right) ,$$

since $\lambda(x_0) + \lambda(y_0) = 2(k+1)\lambda\big((xy)^r x\big) = \lambda(xy)$.

Case 8.32232. If $s = r + 1$ we have, after cancellation,

$$\begin{aligned} W^* = {}& \cdot\bar y\bar x\cdot\{\cdot yx)\big((xy)^r x\big)^{2k+1}(xy\cdot\bar x\bar y)(\bar x\bar y)^{r-1}\bar x \\ & (\bar x\bar y)^s\bar x\big((\bar x\bar y)^r\bar x\big)^{2k}(\bar x\bar y\cdot xy)(xy)^{r-1}x(xy\cdot\bar x\bar y)(\bar x\bar y)^{r-1}\bar x \\ & (\bar y\bar x\cdot yx)\big((xy)^r x\big)^{2k}(xy)^s x(xy)^{r-1}x(yx\cdot\bar y\bar x)\big((\bar x\bar y)^r\bar x\big)^{2k+1}(\bar y\bar x\cdot\}^N\cdot yx\cdot * \ . \end{aligned}$$

The case $r = 0$, $s = 1$ does not arise, as this would imply $x_0 = x$, $y_0 = \bar x^{k+1}\bar y\bar x^{k+1}$, contrary to the definition of k as the largest integer for which $y_0 = \bar x_0{}^k\bar y\bar x_0{}^k$ for some y . Thus $r - 1$ is non-negative, and the above expression makes sense. Just as in Case 8.31232, we observe that

$$\lambda(x_0) + \lambda(y_0) = 2(k+1)\lambda\big((xy)^r x\big) + \lambda(xy) ,$$

and using Theorems 12, 21 and 22 we compute

$$\begin{aligned} \lambda(W^*) &= \lambda(\bar y\bar x\cdot yx\cdot\bar y\bar x\cdot yx) - \lambda(\bar y\bar x\cdot yx) + N\{2\lambda\big((xy)^{r-1}x\big)+(8k+6)\lambda\big((xy)^r x\big) \\ &\qquad +3\lambda(xy\cdot\bar x\bar y)+3\lambda(\bar x\bar y\cdot xy)\} \\ &\geq 4 + 12N + 4N\big(\lambda(x_0)+\lambda(y_0)\big) \ . \end{aligned}$$

This completes our discussion of Case 8.32, and hence of Case 8.

Case 9. Non-trivial cancellation is restricted to the two context (B): $\bar y_0\cdot x_0$ and (D): $x_0\cdot\bar y_0$. Our discussion of this case follows closely the pattern of Case 8. We have

$$W^* = y_0\cdot\bar x_0\{\bar y_0\cdot x_0\cdot\bar y_0\bar x_0{}^2\cdot y_0{}^2x_0{}^2\cdot\bar y_0\bar x_0\cdot y_0\cdot\bar x_0\bar y_0{}^2\cdot x_0{}^2y_0{}^2\cdot\bar x_0\}^{N}\bar y_0\cdot x_0{}^*$$

(here the absence of cancellation-points fore and aft of this expression signifies that no non-trivial cancellation can take place between the x_0 at the "end" and the y_0 at the "beginning").

We consider sub-cases according to whether cancellation dies in $\bar y_0\cdot x_0$, $x_0\cdot\bar y_0$, or neither.

Case 9.1. If cancellation dies in $\bar y_0\cdot x_0$, we transcribe $(x_0, y_0) \to (zx, zy)$, where $x \neq 1$, $y \neq 1$, and $\bar y\cdot x = \bar y x$. Now we have

$$W^* = zy\cdot\bar x\bar z\{\bar y x\cdot\bar y\bar z\bar x)\bar z\bar x yz\big(y(zx)^2\cdot\bar y\bar z\bar x y\cdot\bar x\bar z\bar y\big)\bar z\bar y xz(x(zy)^2\cdot\bar x\bar z\}^{N}\bar y)(x^*$$

where zx, zy are not coradical. By Theorem 20 and Theorem 24, cancellation dies in the bracketed portions, and

$$\begin{aligned} \lambda(W^*) &\geq \lambda(zy) + 4 + (N-1)\big(2\lambda(zy)+4\big) + \lambda(zy) + 4 \\ &\qquad + N\{2\lambda(xz)+2\lambda(yz)+2\lambda(zx)+4\} \\ &= 4 + 8N + 4N\big(\lambda(x_0)+\lambda(y_0)\big) \ . \end{aligned}$$

Case 9.2. If cancellation dies in $x_0\cdot\bar y_0$, we transcribe

$(x_0, y_0) \to (xz, yz)$, where $x \neq 1$, $y \neq 1$, and $x \cdot \bar{y} = x\bar{y}$. We have

$$W^* = y)(\bar{x}\{\bar{z}\bar{y} \cdot x\bar{y}(\bar{z}\bar{x})^2 \cdot (yz)^2x)zx\big(\bar{y}\bar{z}\bar{x} \cdot y\bar{x}(\bar{z}\bar{y})^2 \cdot (xz)^2y\big)zy(\bar{x}\}^N\bar{z}\bar{y} \cdot xz^*$$

where xz, yz are not coradical. By Theorem 20 and Theorem 24, cancellation dies in the bracketed portions, and

$$\begin{aligned}\lambda(W^*) &\geq 4 + N\{\lambda(xz)+2\lambda(yz)+4+\lambda(zx)+\lambda(yz)+2\lambda(xz)+4+\lambda(zy)\} \\ &= 4 + 8N + 4N\big(\lambda(x_0)+\lambda(y_0)\big) .\end{aligned}$$

Case 9.3. If cancellation persists in $\bar{y}_0 \cdot x_0$ and in $x_0 \cdot \bar{y}_0$, by Theorem 9 we may transcribe $(x_0, y_0) \to \big((xy)^m x, (xy)^n x\big)$, where x, y are not coradical and m, n are non-negative integers differing by 1 . We consider separately the cases $m = n + 1$ and $n = m + 1$.

Case 9.31. If $m = n + 1$ we have, after cancellation,

$$W^* = \cdot\bar{y}\bar{x}\cdot\{\cdot y \cdot (\bar{y}\bar{x})^n(\bar{x}\bar{y})^m\bar{x}^2\bar{y} \cdot (xy)^n x(xy)^m x^2y \cdot \bar{x}\bar{y}(\bar{x}\bar{y})^m\bar{x}(\bar{x}\bar{y})^n\bar{x}\cdot$$
$$\cdot yx(xy)^m x(xy)^n x \cdot \bar{y}\bar{x}\cdot\}^N \cdot yx \cdot {}^* .$$

We consider separately the cases $n > 0$, $n = 0$.

Case 9.311. If $n > 0$ we have

$$W^* = \cdot\bar{y}\{\bar{x})\bar{x}(\bar{y}\bar{x})^{n-1}(\bar{x}\bar{y})^m\bar{x}(\bar{x}\bar{y} \cdot xy)(xy)^{n-1}x$$
$$(xy)^m x(xy \cdot \bar{x}\bar{y})(\bar{x}\bar{y})^m\bar{x}(\bar{x}\bar{y})^{n-1}\bar{x}(\bar{y}\bar{x} \cdot yx)(xy)^m x(xy)^{n-1}x(yx \cdot \bar{y}\}^N\bar{x} \cdot yx \cdot {}^*$$

(we have moved the braces to the left of their usual place). By Theorems 12, 21 and 22, cancellation dies in the bracketed portions, and

$$\begin{aligned}\lambda(W^*) &= \lambda(yx \cdot \bar{y}\bar{x} \cdot yx \cdot \bar{y}\bar{x}) - \lambda(yx \cdot \bar{y}\bar{x}) + N\big\{8\lambda\big((xy)^n x\big)+2\lambda(xy \cdot \bar{x}\bar{y})+2\lambda(\bar{x}\bar{y} \cdot xy)\big\} \\ &\geq 4 + N\big\{8\lambda\big((xy)^n x\big)+4\lambda(xy)+8\big\} \\ &= 4 + 8N + 4N\big(\lambda(x_0)+\lambda(y_0)\big) .\end{aligned}$$

Case 9.312. If $n = 0$ and $m = 1$ we have

$$W^* = \cdot\bar{y}\bar{x}\cdot\big\{\cdot y \cdot \bar{x}\bar{y}\bar{x}^2\bar{y} \cdot x^2yx^2y \cdot \bar{x}\bar{y}\bar{x}\bar{y}\bar{x}^2 \cdot yx^2yx^2 \cdot \bar{y}\bar{x}\cdot\big\}^N \cdot yx \cdot {}^*$$

where $x_0 = xyx$, $y_0 = x$. That is, $x_0 = y_0yy_0$. Now we transcribe $y \to x^{k-1}yx^{k-1}$, choosing the positive integer k as large as possible. After this transcription we have

$$W^* = \cdot\bar{y}\bar{x}\cdot\{\cdot y \cdot \bar{x}\bar{y})\bar{x}^{2k-1}(\bar{x}\bar{y} \cdot x^2y)x^{2k-1}(xy \cdot \bar{x}\bar{y})$$
$$\bar{x}^{2k-1}(\bar{y}\bar{x}^2 \cdot yx)x^{2k-1}(yx^2 \cdot \bar{y}\bar{x}\cdot\}^N \cdot yx \cdot {}^*$$

where $x_0 = x^kyx^k$, $y_0 = x$, and x, y are not coradical. We consider separately the cases in which cancellation dies in $\bar{y} \cdot x$, $x \cdot \bar{y}$, or neither.

Case 9.3121. If cancellation dies in $\bar{y} \cdot x$ we may transcribe $(x, y) \to (zx, zy)$. This gives

$$W^* = \bullet\bar{y}\bar{z}\{\bar{x}y\bullet\bar{x}\bar{z}\bar{y})(\bar{z}\bar{x})^{2k}\bar{z}\bar{y}(xz)^2yz(xz)^{2k-1}$$
$$(xzy\bullet\bar{x}\bar{z}\bar{y})(\bar{z}\bar{x})^{2k-1}\bar{z}\bar{y}(\bar{z}\bar{x})^2yz(xz)^{2k}(y(zx)^2\bullet\bar{y}\bar{z}\}^N\bar{x})(yzx\bullet *$$

where $x_0 = (zx)^k zy(zx)^k$, $y_0 = zx$, and zx, zy are not coradical. By Theorems 20 and 24 cancellation dies in the bracketed portions, and

$$\lambda(W^*) \geq \lambda(zx) + 4 + (N-1)\big(2\lambda(zx)+4\big) + \lambda(zx) + 4$$
$$+ N\{\lambda(x_0)+\lambda(y_0)+\lambda(x_0)+4+\lambda(x_0)+\lambda(y_0)+\lambda(x_0)\}$$
$$= 4 + 8N + 4N\big(\lambda(x_0)+\lambda(y_0)\big) .$$

Case 9.3122. If cancellation dies in $x\bullet\bar{y}$ we may transcribe $(x, y) \to (xz, yz)$. This gives

$$W^* = \big\{(\bar{y}\bar{z}\bar{x}\bullet y\bar{x}\bar{z})\bar{y}\bar{z}(\bar{x}\bar{z})^{2k-1}\big(\bar{x}\bar{z}\bar{y}\bullet(xz)^2y\big)(zx)^{2k}zy$$
$$\bar{x}\bar{z}\bar{y}\bar{z}(\bar{x}\bar{z})^{2k-1}\big(\bar{y}(\bar{z}\bar{x})^2\bullet yzx\big)(zx)^{2k-1}zy(zx)^2\big\}^N(\bar{y}\bar{z}\bar{x}\bullet yzx)^*$$

where $x_0 = (xz)^k yz(xz)^k$, $y_0 = xz$, and xz, yz are not coradical. (As is evident, we have moved the braces to the left.) By Theorems 20 and 25, cancellation dies in the bracketed portions. In estimating the length of W^* we consider together the first and second bracketed portions (that is, bracketed portions including cancellation-points) in order to apply Theorem 26. The length is thus at least

$$4 + N\big\{\big(3\lambda(xz)+4\big)+\big(\lambda(x_0)-\lambda(y_0)\big)+2\lambda(x_0)+\big(\lambda(xz)+4\big)+\lambda(x_0)+\lambda(y_0)\big\}$$
$$= 4 + 8N + 4N\big(\lambda(x_0)+\lambda(y_0)\big) .$$

Case 9.3123. If cancellation persists in $\bar{y}\bullet x$ and $x\bullet\bar{y}$, by Theorem 9 we may transcribe $(x, y) \to \big((xy)^r x, (xy)^s x\big)$, where r, s are non-negative integers differing by 1 . After this transcription we have

$$x_0 = \big((xy)^r x\big)^k (xy)^s x\big((xy)^r x\big)^k \ , \quad y_0 = (xy)^r x \ ,$$

where x, y are not coradical. We consider separately the cases $r = s + 1$ and $s = r + 1$.

Case 9.31231. If $r = s + 1$ we have, after cancellation,

$$W^* = \bullet\bar{x}\{\bar{y})(\bar{x}\bar{y})^r\bar{x}(\bar{x}\bar{y})^s\bar{x}\big((\bar{x}\bar{y})^r\bar{x}\big)^{2k-1}(\bar{x}\bar{y})^s\bar{x}$$
$$(\bar{y}\bar{x}\bullet yx)(xy)^r x(xy)^s x\big((xy)^r x\big)^{2k-1}(xy)^s x(yx\bullet\bar{y}\bar{x})$$
$$(\bar{x}\bar{y})^s\bar{x}\big((\bar{x}\bar{y})^r\bar{x}\big)^{2k-1}(\bar{x}\bar{y})^s\bar{x}(\bar{x}\bar{y})^r\bar{x}(\bar{x}\bar{y}\bullet xy)(xy)^s x$$
$$\big((xy)^r x\big)^{2k-1}(xy)^s x(xy)^r x(xy\bullet\bar{x}\}^N\bar{y}\bullet xy\bullet * \ .$$

Cancellation dies in the bracketed portions, and by Theorems 21, 22 we have

$$\lambda(W^*) = \lambda(xy\cdot\bar{x}\bar{y}\cdot xy\cdot\bar{x}\bar{y}) - \lambda(xy\cdot\bar{x}\bar{y})$$
$$+ N\{8(k+1)\lambda\big((xy)^r x\big)-8\lambda(xy)+2\lambda(xy\cdot\bar{x}\bar{y})+2\lambda(\bar{x}\bar{y}\cdot xy)\}$$
$$\geq 4 + N\{8(k+1)\lambda\big((xy)^r x\big)-8\lambda(xy)+2\big(2\lambda(xy)+4\big)\}$$
$$= 4 + 8N + 4N\big(\lambda(x_0)+\lambda(y_0)\big) .$$

Case 9.31232. If $s = r + 1$ we have, after cancellation,

$$W^* = \cdot\bar{y}\{\bar{x})(\bar{x}\bar{y})^{r-1}\bar{x}(\bar{x}\bar{y})^s\bar{x}\big((\bar{x}\bar{y})^r\bar{x}\big)^{2k}$$
$$(\bar{x}\bar{y}\cdot xy)(xy)^{r-1}x(xy)^s x\big((xy)^r x\big)^{2k}(xy\cdot\bar{x}\bar{y})\big((\bar{x}\bar{y})^r\bar{x}\big)^{2k}(\bar{x}\bar{y})^s\bar{x}(\bar{x}\bar{y})^{r-1}\bar{x}$$
$$(\bar{y}\bar{x}\cdot yx)\big((xy)^r x\big)^{2k}(xy)^s x(xy)^{r-1}x(yx\cdot\bar{y}\}^N\bar{x}\cdot yx\cdot * .$$

The case $r = 0$, $s = 1$ does not arise, as this would imply $x_0 = x^{k+1}yx^{k+1}$, $y_0 = x$, contrary to the definition of k as the largest integer such that $x_0 = y_0{}^k y y_0{}^k$ for some y . Thus $r-1 \geq 0$, and the above expression is legitimate.

By Theorems 12, 21, and 22, we have

$$\lambda(W^*) = \lambda(yx\cdot\bar{y}\bar{x}\cdot yx\cdot\bar{y}\bar{x}) - \lambda(yx\cdot\bar{y}\bar{x})$$
$$+ N\{8(k+1)\lambda\big((xy)^r x\big)+2\lambda(yx\cdot\bar{y}\bar{x})+2\lambda(\bar{y}\bar{x}\cdot yx)\}$$
$$\geq 4 + N\{8(k+1)\lambda\big((xy)^r x\big)+2\big(2\lambda(xy)+4\big)\}$$
$$= 4 + 8N + 4N\big(\lambda(x_0)+\lambda(y_0)\big) .$$

This disposes of Case 9.31.

Case 9.32. If $n = m + 1$ we have, after cancellation,

$$W^* = \cdot xy\cdot\{\cdot\bar{x}\bar{y}(\bar{x}\bar{y})^n\bar{x}(\bar{x}\bar{y})^m\bar{x}\cdot yx(xy)^n x(xy)^m x\cdot$$
$$\cdot\bar{y}\bar{x}\cdot yx\cdot(\bar{x}\bar{y})^m\bar{x}(\bar{x}\bar{y})^n\bar{x}\bar{x}\bar{y}\cdot(xy)^m x(xy)^n xxy\cdot\}^N\cdot\bar{x}\bar{y}\cdot * .$$

We consider separately the cases $m > 0$, $m = 0$.

Case 9.321. If $m > 0$ we have

$$W^* = \cdot xy\cdot\{\cdot\bar{x}\bar{y})(\bar{x}\bar{y})^n\bar{x}(\bar{x}\bar{y})^{m-1}\bar{x}(\bar{y}\bar{x}\cdot yx)(xy)^n x(xy)^{m-1}x$$
$$(yx\cdot\bar{y}\bar{x})(\bar{x}\bar{y})^{m-1}\bar{x}(\bar{x}\bar{y})^n\bar{x}(\bar{x}\bar{y}\cdot xy)(xy)^{m-1}x(xy)^n x(xy\cdot\}^N\cdot\bar{x}\bar{y}\cdot * .$$

By Theorems 21 and 22,

$$\lambda(W^*) = \lambda(xy\cdot\bar{x}\bar{y}\cdot xy\cdot\bar{x}\bar{y}) - \lambda(xy\cdot\bar{x}\bar{y}) + N\{8\lambda\big((xy)^m x\big)+2\lambda(xy\cdot\bar{x}\bar{y})+2\lambda(\bar{x}\bar{y}\cdot xy)\}$$
$$\geq 4 + N\{8\lambda\big((xy)^m x\big)+2\big(2\lambda(xy)+4\big)\}$$
$$= 4 + 8N + 4N\big(\lambda(x_0)+\lambda(y_0)\big) .$$

Case 9.322. If $m = 0$, $n = 1$ we have

$$W^* = \cdot xy\cdot\{\cdot\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}x^2\cdot yx^2yx^2\cdot\bar{y}\bar{x}\cdot y\cdot\bar{x}\bar{y}\bar{x}x^2\bar{y}\cdot x^2yx^2y\cdot\}^N\cdot\bar{x}\bar{y}\cdot *$$

where $x_0 = x$, $y_0 = xyx$. That is, $y_0 = x_0 y x_0$. Now we transcribe $y \to x^{k-1}yx^{k-1}$, choosing the positive integer k as large as possible. After this transcription we have

$$W^* = \cdot xy\cdot\{\cdot\bar{x}\bar{y})\bar{x}^{2k-1}(\bar{y}\bar{x}^2\cdot yx)x^{2k-1}$$
$$(yx^2\cdot\bar{y}\bar{x}\cdot y\cdot\bar{x}\bar{y})\bar{x}^{2k-1}(\bar{x}\bar{y}\cdot x^2y)x^{2k-1}(xy\cdot\}^N\cdot\bar{x}\bar{y}\cdot *$$

where $x_0 = x$, $y_0 = x^kyx^k$, and x, y are not coradical. We consider separately the cases in which cancellation dies in $\bar{y}\cdot x$, $x\cdot\bar{y}$, or neither.

Case 9.3221. If cancellation dies in $\bar{y}\cdot x$ we may transcribe $(x, y) \to (zx, zy)$. This gives

$$W^* = (xzy\cdot\{\cdot\bar{x}\bar{z}\bar{y})(\bar{z}\bar{x})^{2k-1}\bar{z}\bar{y}(\bar{z}\bar{x})^2yz(xz)^{2k}$$
$$\big(y(zx)^2\cdot\bar{y}\bar{z}\bar{x}y\cdot\bar{x}\bar{z}\bar{y}\big)(\bar{z}\bar{x})^{2k}\bar{z}\bar{y}(xz)^2yz(xz)^{2k-1}(xzy\cdot\}^N\cdot\bar{x}\bar{z}\bar{y})^*$$

where $x_0 = zx$, $y_0 = (zx)^kzy(zx)^k$, and zx, zy are not coradical. By Theorems 20 and 24, we have

$$\lambda(W^*) \geq 4 + N\{\lambda(y_0)+\lambda(x_0)+\lambda(y_0)+2\lambda(zx)+4+\lambda(y_0)+\lambda(x_0)+\lambda(y_0)+4\}$$
$$= 4 + 8N + 4N\big(\lambda(x_0)+\lambda(y_0)\big) .$$

Case 9.3222. If cancellation dies in $x\cdot\bar{y}$ we may transcribe $(x, y) \to (xz, yz)$. This gives

$$W^* = \cdot xzy)\{\bar{x}\bar{z}\bar{y}\bar{z}(\bar{x}\bar{z})^{2k-1}\big(\bar{y}(\bar{z}\bar{x})^2\cdot yzx\big)(zx)^{2k-1}zy$$
$$(zx)^2(\bar{y}\bar{z}\bar{x}\cdot y\bar{x}\bar{z})\bar{y}\bar{z}(\bar{x}\bar{z})^{2k-1}\big(\bar{x}\bar{z}\bar{y}\cdot(xz)^2y\big)(zx)^{2k}zy\}^N(\bar{x}\bar{z}\bar{y}\cdot *$$

where $x_0 = xz$, $y_0 = (xz)^kyz(xz)^k$, and xz, yz are not coradical. By Theorems 20 and 25, cancellation dies in the bracketed portions. Within the braces we consider together the first and second bracketed portions (including cancellation-points) in order to apply Theorem 26. We have then

$$\lambda(W^*) \geq 4 + N\{\lambda(y_0)+\big(3\lambda(xz)+4\big)+\lambda(y_0)+\lambda(x_0)+\big(\lambda(y_0)-\lambda(zx)\big)+\lambda(xz)+4+\lambda(y_0)\}$$
$$= 4 + 8N + 4N\big(\lambda(x_0)+\lambda(y_0)\big) .$$

Case 9.3223. If cancellation persists in $\bar{y}\cdot x$ and in $x\cdot\bar{y}$, by Theorem 9 we may transcribe $(x, y) \to \big((xy)^rx, (xy)^sx\big)$, where r, s are non-negative integers differing by 1 . After this transcription we have

$$x_0 = (xy)^rx , \quad y_0 = \big((xy)^rx\big)^k(xy)^sx\big((xy)^rx\big)^k ,$$

where x, y are not coradical. We consider separately the cases $r = s + 1$ and $s = r + 1$.

Case 9.32231. If $r = s + 1$ we have, after cancellation,

$$\begin{aligned} W^* = \bullet yx \bullet \{ \bullet \bar{y}\bar{x})(\bar{x}\bar{y})^s \bar{x}((\bar{x}\bar{y})^r \bar{x})^{2k-1}(\bar{x}\bar{y})^s \bar{x}(\bar{x}\bar{y})^r \bar{x} \\ (\bar{x}\bar{y} \bullet xy)(xy)^s x((xy)^r x)^{2k-1}(xy)^s x(xy)^r x \\ (xy \bullet \bar{x}\bar{y})(\bar{x}\bar{y})^r \bar{x}(\bar{x}\bar{y})^s \bar{x}((\bar{x}\bar{y})^r \bar{x})^{2k-1}(\bar{x}\bar{y})^s \bar{x} \\ (\bar{y}\bar{x} \bullet yx)(xy)^r x(xy)^s x((xy)^r x)^{2k-1}(xy)^s x(yx \bullet \}^N \bullet \bar{y}\bar{x} \bullet * \ . \end{aligned}$$

By Theorems 12, 21 and 22, cancellation dies in the bracketed portions, and we have

$$\begin{aligned} \lambda(W^*) &= \lambda(yx \bullet \bar{y}\bar{x} \bullet yx \bullet \bar{y}\bar{x}) - \lambda(yx \bullet \bar{y}\bar{x}) \\ &\qquad + N\{8(k+1)\lambda((xy)^r x) - 8\lambda(xy) + 2\lambda(yx \bullet \bar{y}\bar{x}) + 2\lambda(\bar{y}\bar{x} \bullet yx)\} \\ &\geq 4 + N\{8(k+1)\lambda((xy)^r x) - 8\lambda(xy) + 2(2\lambda(xy)+4)\} \\ &= 4 + 8N + 4N\{2(k+1)\lambda((xy)^r x) - \lambda(xy)\} \\ &= 4 + 8N + 4N(\lambda(x_0)+\lambda(y_0)) \ . \end{aligned}$$

Case 9.32232. If $s = r + 1$ we have, after cancellation,

$$\begin{aligned} W^* = \bullet xy \bullet \{ \bullet \bar{x}\bar{y})((\bar{x}\bar{y})^r \bar{x})^{2k}(\bar{x}\bar{y})^s \bar{x}(\bar{x}\bar{y})^{r-1}\bar{x} \\ (\bar{y}\bar{x} \bullet yx)((xy)^r x)^{2k}(xy)^s x(xy)^{r-1}x(yx \bullet \bar{y}\bar{x})(\bar{x}\bar{y})^{r-1}\bar{x} \\ (\bar{x}\bar{y})^s \bar{x}((\bar{x}\bar{y})^r \bar{x})^{2k}(\bar{x}\bar{y} \bullet xy)(xy)^{r-1}x(xy)^s x((xy)^r x)^{2k}(xy \bullet \}^N \bullet \bar{x}\bar{y} \bullet * \ . \end{aligned}$$

The case $r = 0$, $s = 1$ does not arise, as this would imply $x_0 = x$, $y_0 = x^{k+1}yx^{k+1}$, contradicting the definition of k as the largest integer such that $y_0 = x_0{}^k y x_0{}^k$ for some y . Thus $r-1 \geq 0$, and the above expression is legitimate.

By Theorems 12, 21 and 22, the length of W^* is at least

$$\begin{aligned} \lambda(xy \bullet \bar{x}\bar{y} \bullet xy \bullet \bar{x}\bar{y}) - \lambda(xy \bullet \bar{x}\bar{y}) + N\{8(k+1)\lambda((xy)^r x) + 2\lambda(xy \bullet \bar{x}\bar{y}) + 2\lambda(\bar{x}\bar{y} \bullet xy)\} \\ \geq 4 + N\{8(k+1)\lambda((xy)^r x) + 2(2\lambda(xy)+4)\} \\ = 4 + 8N + 4N(\lambda(x_0)+\lambda(y_0)) \ . \end{aligned}$$

This completes our discussion of Case 9.

Case 10. Non-trivial cancellation is restricted to the three contexts (A): $y_0 \bullet x_0$, (D): $x_0 \bullet \bar{y}_0$, and (E): $x_0 \bullet x_0$. We have

$$W^* = y_0 \bullet \bar{x}_0 \bullet \{ \bullet \bar{y}_0 x_0 \bullet \bar{y}_0 \bar{x}_0 \bullet \bar{x}_0 y_0{}^2 \bullet x_0 \bullet x_0 \bullet \bar{y}_0 \bar{x}_0 y_0 \bullet \bar{x}_0 \bullet \bar{y}_0{}^2 x_0 \bullet x_0 y_0{}^2 \bullet \bar{x}_0 \bullet \}^N \bullet \bar{y}_0 x_0 *$$

and we must consider separately whether cancellation dies in $y_0 \bullet x_0$, in $x_0 \bullet \bar{y}_0$, or in neither of these contexts.

The third of these possibilities may be eliminated forthwith. For, if cancellation persists in $y_0 \bullet x_0$ and in $x_0 \bullet \bar{y}_0$ then by Theorem 8 we may transcribe $x_0 \to \bar{y}_0 x y_0$. Let us now transcribe $x_0 \to \bar{y}_0{}^k x_1 y_0{}^k$, choosing the positive integer k as large as possible. Clearly (x_1, y_0) is the image of (x_0, y_0) under conjugation by $y_0{}^k$, and x_1, y_0 are not coradical. We have

$W(x_0, y_0) = \bar{y}_0{}^k \bullet W(x_1, y_0) \bullet y_0{}^k$, and

$$W^* = y_0 \cdot \bar{x}_1 \cdot \{\cdot \bar{y}_0 x_1 \cdot \bar{y}_0 \bar{x}_1 \cdot \bar{x}_1 y_0{}^2 \cdot x_1 \cdot x_1 \cdot \bar{y}_0 \bar{x}_1 y_0 \cdot \bar{x}_1 \cdot \bar{y}_0{}^2 x_1 \cdot x_1 y_0{}^2 \cdot \bar{x}_1 \cdot\}^N \cdot \bar{y}_0 x_1{}^*$$

which is exactly the same as the previous expression for W^* in terms of x_0 and y_0, except for the replacement of x_0 by x_1. (It is not hard to see that the same would be true for any 2-variable word W, and not just for the particular word that we have chosen to study.) But now cancellation must die either in $y_0 \cdot x_1$ or in $x_1 \cdot \bar{y}_0$, otherwise a further transcription $x_1 \to \bar{y}_0 x y_0$ would be possible, contrary to the definition of k as the largest possible integer for which $x_0 = \bar{y}_0{}^k x_1 y_0{}^k$ for some x_1. Thus, relabelling if necessary, we may assume that cancellation dies either in $y_0 \cdot x_0$ or in $x_0 \cdot \bar{y}_0$.

Case 10.1. If cancellation dies in $y_0 \cdot x_0$, we transcribe $(x_0, y_0) \to (zx, y\bar{z})$, where $x \neq 1$, $y \neq 1$, and $y \cdot x = yx$. This transcription gives

$$W^* = y\bar{z} \cdot \bar{x}\{\bar{y}zx \cdot z\bar{y}\bar{x}\bar{z} \cdot \bar{x}\bar{z}y\bar{z}yx \cdot zx \cdot z\bar{y}\bar{x}\bar{z}y\bar{z} \cdot \bar{x}\bar{y}z\bar{y}zx \cdot zxy\bar{z}y\bar{z} \cdot \bar{x}\}^N \bar{y}zx^*$$

where zx, $y\bar{z}$ are not coradical. Now we consider separately three cases: one in which cancellation dies in $x \cdot z$, and two in which it persists, corresponding to the two transcriptions $x \to x\bar{z}$ and $z \to \bar{x}z$, at least one of which is possible if cancellation persists in $x \cdot z$.

Case 10.11. If cancellation dies in $x \cdot z$, then we transcribe $(x, z) \to (xu, \bar{u}z)$ so as to give

$$W^* = y\bar{z}\bar{x}\{\bar{y}\bar{u}zxz\bar{y}\bar{u}\bar{x}\bar{z}\bar{x}\bar{z}uy\bar{z}uyxzxz\bar{y}\bar{u}\bar{x}\bar{z}uy\bar{z}\bar{x}\bar{y}\bar{u}z\bar{y}\bar{u}zxzxuy\bar{z}uy\bar{z}\bar{x}\}^N \bar{y}\bar{u}zxu^*$$

where $x_0 = \bar{u}zxu$, $y_0 = y\bar{z}u$, $x \neq 1$, $y \neq 1$, $z \neq 1$, and $x \cdot z = xz$. Now uy is reduced, so we have $W(x_0, y_0) = \bar{u} \cdot W(x_1, y_1) \cdot u$, where (x_1, y_1) is the image of (x_0, y_0) under conjugation by u, that is, $x_1 = zx$ and $y_1 = uy\bar{z}$. The expression for W^* is reduced, so we calculate directly

$$\begin{aligned} \lambda(W^*) &= 2\big(\lambda(x)+\lambda(uy)+\lambda(z)\big) + N\{10\lambda(x)+10\lambda(uy)+14\lambda(z)\} \\ &= (2+6N)\big(\lambda(x)+\lambda(uy)+\lambda(z)\big) + 4N\big(\lambda(zx)+\lambda(uy\bar{z})\big) \\ &\geq 6 + 18N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) . \end{aligned}$$

Case 10.12. The transcription $x \to x\bar{z}$ gives

$$W^* = y \cdot \bar{x}\{\bar{y}zx \cdot \bar{y}z\bar{x} \cdot \bar{x}\bar{z}y\bar{z}yx \cdot x \cdot \bar{y}z\bar{x}\bar{z}y \cdot \bar{x}\bar{y}z\bar{y}zx \cdot x\bar{z}y\bar{z}y \cdot \bar{x}\}^N \bar{y}zx\bar{z}^*$$

where $x_0 = zx\bar{z}$, $y_0 = y\bar{z}$, and $y \neq 1$. By Theorem 7, x and $\bar{z}y$ are not coradical. (Note that $\bar{z}y$ appears as a reduced word in the above expression.) Now we have $W(x_0, y_0) = z \cdot W(x_1, y_1) \cdot \bar{z}$, where

(x_1, y_1) is the image of (x_0, y_0) under conjugation by $\bar{z}$, that is, $x_1 = x$, $y_1 = \bar{z}y$. We consider separately the cases in which $x = x_1$ is or is not cyclically reduced.

Case 10.121. If x is not cyclically reduced, then since yx appears as a reduced word in the above expression we see that no non-trivial cancellation can take place in $x \cdot \bar{y}$ (recall that we have $y \neq 1$ as well as $x \neq 1$). Thus the expression becomes

$$W^* = y\bar{x}\{\bar{y}zx\bar{y}z(\bar{x}\cdot\bar{x})(\bar{z}y)^2(x\cdot x)\bar{y}z\bar{x}\bar{z}y\bar{x}(\bar{y}z)^2(x\cdot x)(\bar{z}y)^2\bar{x}\}^N\bar{y}zx\bar{z}^*$$

and we have

$$\begin{aligned}\lambda(W^*) &\geq 2\big(\lambda(x)+\lambda(y)+\lambda(z)\big) + N\{7\lambda(x)+10\lambda(y)+10\lambda(z)+3\} \\ &= (2+4N)\big(\lambda(x_1)+\lambda(y_1)\big) + 3N + 3N\big(\lambda(x_1)+2\lambda(y_1)\big) \\ &\geq 4 + 12N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) .\end{aligned}$$

Case 10.122. If x is cyclically reduced, we have

$$W^* = y\cdot\bar{x}\{\bar{y}zx\cdot\bar{y}z\bar{x})\bar{x}\bar{z}y(\bar{z}yx^2\cdot\bar{y}z\bar{x}\bar{z}y\cdot\bar{x}\bar{y}z)\bar{y}zx(x(\bar{z}y)^2\cdot\bar{x}\}^N\bar{y}z)(x\bar{z}^*$$

and by Theorem 13 and Theorem 19 we have

$$\begin{aligned}\lambda(W^*) &= \lambda(x\bar{z}y\cdot\bar{x}\bar{y}zx\cdot\bar{y}z\bar{x}) + \lambda\big(x(\bar{z}y)^2\cdot\bar{x}\bar{y}z\big) + (N-1)\lambda\big(x(\bar{z}y)^2\cdot\bar{x}\bar{y}zx\cdot\bar{y}z\bar{x}\big) \\ &\qquad + N\{2\big(\lambda(x)+\lambda(y)+\lambda(z)\big)+\lambda(\bar{z}yx^2\cdot\bar{y}z\bar{x}\bar{z}y\cdot\bar{x}\bar{y}z)\} \\ &\geq 4 + 8N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) .\end{aligned}$$

Case 10.13. The transcription $z \to \bar{x}z$ gives

$$W^* = y\bar{z}\cdot\{\cdot\bar{y}\bar{x}z\cdot z\bar{y}\bar{x}\bar{z}\cdot\bar{z}xy\bar{z}xy\cdot z\cdot z\bar{y}\bar{x}\bar{z}xy\bar{z}\cdot\bar{y}\bar{x}\bar{z}\bar{y}\bar{x}z\cdot zxy\bar{z}xy\bar{z}\cdot\}^N\cdot\bar{y}\bar{x}zx^*$$

where $x_0 = \bar{x}zx$, $y_0 = y\bar{z}x$, $x \neq 1$, $y \neq 1$. Since xy appears as a reduced word in the above expression, Theorem 7 implies that z and $xy\bar{z}$ are not coradical. Now $W(x_0, y_0) = \bar{x}\cdot W(x_1, y_1)\cdot x$, where (x_1, y_1) is the image of (x_0, y_0) under conjugation by $\bar{x}$, that is, $x_1 = z$, $y_1 = xy\bar{z}$.

We consider separately the cases in which z is or is not cyclically reduced.

Case 10.131. If z is not cyclically reduced, then no non-trivial cancellation can take place in $y \cdot z$, since $z\bar{y}$ appears as a reduced word. The expression becomes

$$W^* = y\bar{z}\{\bar{y}\bar{x}(z\cdot z)\bar{y}\bar{x}(\bar{z}\cdot\bar{z})xy\bar{z}xy(z\cdot z)\bar{y}\bar{x}\bar{z}xy\bar{z}\bar{y}\bar{x}z\bar{y}\bar{x}(z\cdot z)xy\bar{z}xy\bar{z}\}^N\bar{y}\bar{x}zx^*$$

and we have

$$\begin{aligned}\lambda(W^*) &\geq 2\big(\lambda(x)+\lambda(y)+\lambda(z)\big) + N\{10\lambda(x)+10\lambda(y)+10\lambda(z)+4\} \\ &\geq 6 + 18N + 4N\big(\lambda(x_1)+\lambda(y_1)\big)\end{aligned}$$

since $\lambda(x) \geq 1$, $\lambda(y) \geq 1$, $\lambda(z) \geq 1$.

Case 10.132. If z is cyclically reduced, then we have

$$W^* = y\bar{z}\cdot\{\cdot\bar{y}\bar{x}z)z\bar{y}\bar{x}\bar{z}^2xy(\bar{z}xy\cdot z^2\bar{y}\bar{x})\bar{z}(xy\bar{z}\cdot\bar{y}\bar{x}z)\bar{y}\bar{x}z^2xy\bar{z}(xy\bar{z}\cdot\}^N\cdot\bar{y}\bar{x}z)(x^*$$

and by Theorem 13 we have

$$\begin{aligned}\lambda(W^*) &\geq 4 + N\{4\lambda(x)+4\lambda(y)+8\lambda(z)+12\} \\ &= 4 + 12N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) .\end{aligned}$$

Case 10.2. If cancellation dies in $x_0\cdot\bar{y}_0$, we transcribe $(x_0, y_0) \to (xz, yz)$, where $x \neq 1$, $y \neq 1$, and $x\cdot\bar{y} = x\bar{y}$. This transcription gives

$$W^* = y\bar{x}\cdot\{\cdot\bar{z}\bar{y}x\bar{y}\bar{z}\bar{x}\cdot\bar{z}\bar{x}yzyz\cdot xz\cdot x\bar{y}\bar{z}\bar{x}y\bar{x}\cdot\bar{z}\bar{y}\bar{z}\bar{y}xz\cdot xzyzy\bar{x}\cdot\}^N\cdot\bar{z}\bar{y}xz^*$$

where xz, yz are not coradical. Now, either cancellation dies in $z\cdot x$, or the expression admits the transcription $x \to \bar{z}x$, or else it admits the transcription $z \to z\bar{x}$. We consider these three cases separately.

Case 10.21. If cancellation dies in $z\cdot x$, then we transcribe $(x, z) \to (ux, z\bar{u})$ so as to give

$$W^* = y\bar{x}\{\bar{z}\bar{y}ux\bar{y}u\bar{z}\bar{x}\bar{z}\bar{x}\bar{u}yz\bar{u}y(zx)^2\bar{y}u\bar{z}\bar{x}\bar{u}y\bar{x}\bar{z}\bar{y}u\bar{z}\bar{y}uxzxz\bar{u}yz\bar{u}y\bar{x}\}^N\bar{z}\bar{y}uxz\bar{u}^*$$

where $x_0 = uxz\bar{u}$, $y_0 = yz\bar{u}$, $z \neq 1$, $x \neq 1$, and $y \neq 1$. We note that $\bar{u}y$ is reduced, and now $W(x_0, y_0) = u\cdot W(x_1, y_1)\cdot\bar{u}$, where (x_1, y_1) is the image of (x_0, y_0) under conjugation by $\bar{u}$, that is, $x_1 = xz$, $y_1 = \bar{u}yz$. We have

$$\begin{aligned}\lambda(W^*) &= 2\big(\lambda(x)+\lambda(\bar{u}y)+\lambda(z)\big) + N\{10\lambda(x)+10\lambda(\bar{u}y)+12\lambda(z)\} \\ &\geq 6 + 16N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) ,\end{aligned}$$

since $x \neq 1$, $\bar{u}y \neq 1$, and $z \neq 1$.

Case 10.22. The transcription $x \to \bar{z}x$ gives

$$W^* = y\bar{x}\cdot\{\cdot\bar{y}\bar{z}x\bar{y}\bar{z}\bar{x}\cdot\bar{x}zyzy\cdot x\cdot x\bar{y}\bar{z}\bar{x}zy\bar{x}\cdot\bar{y}z\bar{y}\bar{z}\bar{x}\cdot xzyzy\bar{x}\cdot\}^N\cdot\bar{y}\bar{z}xz^*$$

where $x_0 = \bar{z}xz$, $y_0 = yz$, and $y \neq 1$. By Theorem 7, since zy is reduced, x and zy are not coradical. Here $(x_1, y_1) = (x, zy)$ is the image of (x_0, y_0) under conjugation by z .

We consider separately the cases in which x is or is not cyclically reduced.

Case 10.221. If x is not cyclically reduced then no non-trivial cancellation can take place in $y\cdot x$, since $y \neq 1$ and $x\bar{y}$ is reduced. Using Corollary 11.1 we see that

$$\begin{aligned}\lambda(W^*) &\geq 2\big(\lambda(x)+\lambda(y)+\lambda(z)\big) + N\{7\lambda(x)+10\lambda(y)+10\lambda(z)+3\} \\ &\geq 4 + 12N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) \ .\end{aligned}$$

Case 10.222. If x is cyclically reduced, we have

$$W^* = y\bar{x}\cdot\{\cdot\bar{y}\bar{z}x)\bar{y}\bar{z}\bar{x}\big(\bar{x}(zy)^2\cdot x^2\bar{y}\bar{z}\big)\bar{x}\big(zy\bar{x}\cdot(\bar{y}\bar{z})^2x\big)xzy(zy\bar{x}\cdot\}^N\cdot\bar{y}\bar{z}x)(z^*$$

and by Theorem 13 we have

$$\begin{aligned}\lambda(W^*) &\geq 4 + N\{3\lambda(x)+2\lambda(zy)+\big(\lambda(x)+\lambda(zy)+4\big)+\big(\lambda(zy)+4\big)+4\} \\ &= 4 + 12N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) \ .\end{aligned}$$

Case 10.23. The transcription $z \to z\bar{x}$ gives

$$W^* = y\cdot\{\cdot\bar{z}\bar{y}x\bar{y}x\bar{z}\cdot\bar{z}\bar{x}yz\bar{x}yz\cdot z\cdot\bar{y}x\bar{z}\bar{x}y\cdot\bar{z}\bar{y}x\bar{z}\bar{y}xz\cdot z\bar{x}yz\bar{x}y\cdot\}^N\cdot\bar{z}\bar{y}xz\bar{x}^*$$

where $x_0 = xz\bar{x}$, $y_0 = yz\bar{x}$, $x \neq 1$, $y \neq 1$. Since $\bar{x}y$ is reduced, the image of (x_0, y_0) under conjugation by x is (x_1, y_1) , given by $x_1 = z$, $y_1 = \bar{x}yz$, and by Theorem 7, x_1 and y_1 are not coradical (and in particular we have $z \neq 1$).

We consider separately whether or not z is cyclically reduced.

Case 10.231. If z is not cyclically reduced, then non-trivial cancellation is ruled out in $z\cdot\bar{y}$, since yz is reduced. Therefore, by Corollary 11.1, the length of W^* is at least

$$\begin{aligned}&2\big(\lambda(x)+\lambda(y)+\lambda(z)\big) + N\{10\lambda(x)+10\lambda(y)+9\lambda(z)+3\} \\ &\qquad\qquad \geq 6 + 16N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) \ .\end{aligned}$$

Case 10.232. If z is cyclically reduced, we have

$$W^* = y\cdot\{\cdot\bar{z}\bar{y}x)\bar{y}x\bar{z}^2\bar{x}yz(\bar{x}yz^2\cdot\bar{y}x\bar{z}\bar{x}y\cdot\bar{z}\bar{y}x)\bar{z}\bar{y}xz^2\bar{x}y(z\bar{x}y\cdot\}^N\cdot\bar{z}\bar{y}x)(z\bar{x}^*$$

and since $z, \bar{x}yz$ are not coradical we have $z, \bar{x}y$ not coradical, by Corollary 4.1. But now by Theorem 13 and Corollary 24.1 we have

$$\begin{aligned}\lambda(W^*) &\geq 4 + N\{2\lambda(\bar{y}x\bar{z}^2\bar{x}yz)+2\lambda(z)+4+4\} \\ &= 4 + 8N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) \ .\end{aligned}$$

This completes our discussion of Case 10.

Case 11. Non-trivial cancellation is restricted to the three contexts (B): $\bar{y}_0\cdot x_0$, (C): $x_0\cdot y_0$, and (E): $x_0\cdot x_0$. We have

$$W^* = \cdot y_0\bar{x}_0\{\bar{y}_0\cdot x_0\bar{y}_0\cdot\bar{x}_0\cdot\bar{x}_0\cdot y_0{}^2x_0\cdot x_0\bar{y}_0\cdot\bar{x}_0\cdot y_0\bar{x}_0\bar{y}_0{}^2\cdot x_0\cdot x_0\cdot y_0{}^2\bar{x}_0\}^N\bar{y}_0\cdot x_0\cdot{}^*$$

and we must consider whether cancellation dies in $\bar{y}_0\cdot x_0$, in $x_0\cdot y_0$, or in neither of these contexts. As in our discussion of Case 10, the third possibility may be eliminated: if cancellation persists in $\bar{y}_0\cdot x_0$ and in $x_0\cdot y_0$ then we may transcribe $x \to y_0{}^k x_1\bar{y}_0{}^k$, where x_1, y_0 are not coradical and k is a positive integer which we may choose as large as possible, and now (x_0, y_0) is the image of

$(x_0,\ y_0)$ under conjugation by $\bar{y}_0^{\,k}$, so that $W(x_0,\ y_0) = y_0{}^k \cdot W(x_1,\ y_0) \cdot \bar{y}^k$, and in the above expression for the circular word W^* we may replace x_0 throughout by x_1 . If in the new expression for W^* cancellation dies neither in $\bar{y}_0 \cdot x_1$ nor in $x_1 \cdot y_0$ then we may transcribe $x_1 \to y_0 x \bar{y}_0$ for some x , contrary to our choice of k to be as large as possible. Thus, relabelling if necessary, we may assume that cancellation dies in at least one of the contexts $\bar{y}_0 \cdot x_0$, $x_0 \cdot y_0$.

Case 11.1. If cancellation dies in $\bar{y}_0 \cdot x_0$, we transcribe $(x_0,\ y_0) \to (zx,\ zy)$, where $x \neq 1$, $y \neq 1$, and $\bar{y} \cdot x = \bar{y}x$. This transcription gives

$$W^* = \cdot zy\bar{x}\bar{z}\{\bar{y}x\bar{y}\bar{z}\cdot\bar{x}\bar{z}\cdot\bar{x}yzyzx\cdot zx\bar{y}\bar{z}\cdot\bar{x}y\bar{x}\bar{z}\bar{y}\bar{z}\bar{y}x\cdot zx\cdot zyzy\bar{x}\bar{z}\}^{N-}\bar{y}x\cdot * \ .$$

We consider separately the cases in which cancellation in $x \cdot z$ dies, persists via the transcription $x \to x\bar{z}$, or persists via the transcription $z \to \bar{x}z$.

Case 11.11. If cancellation dies in $x \cdot z$, we transcribe $(x,\ z) \to (x\bar{u},\ \bar{u}z)$ so as to give

$$W^* = zyu\bar{x}\bar{z}\bar{u}\{\bar{y}x\bar{u}\bar{y}(\bar{z}\bar{x})^2(yuz)^2xzx\bar{u}\bar{y}\bar{z}\bar{x}yu\bar{x}(\bar{z}\bar{u}\bar{y})^2(xz)^2yuzyu\bar{x}\bar{z}\bar{u}\}^{N-}\bar{y}x^*$$

where $x_0 = uzx\bar{u}$, $y_0 = uzy$, $x \neq 1$, $z \neq 1$, and $y \neq 1$. Since yu is reduced, so is zyu . Now $(x_1,\ y_1)$, defined by $x_1 = zx$, $y_1 = zyu$, is the image of $(x_0,\ y_0)$ under conjugation by u . So the length of W^* is

$$2\big(\lambda(x)+\lambda(yu)+\lambda(z)\big) + N\{10\lambda(x)+10\lambda(yu)+12\lambda(z)\}$$
$$\geq 6 + 16N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) \ ,$$

since $x \neq 1$, $yu \neq 1$, and $z \neq 1$.

Case 11.12. The transcription $x \to x\bar{z}$ gives

$$W^* = \cdot yz\bar{x}\bar{z}\{\bar{y}x\bar{z}\bar{y}\cdot\bar{x}\cdot\bar{x}(yz)^2x\cdot x\bar{z}\bar{y}\cdot\bar{x}yz\bar{x}(\bar{z}\bar{y})^2x\cdot x\cdot(yz)^2\bar{x}\bar{z}\}^{N-}\bar{y}x\cdot *$$

where $x_0 = zx\bar{z}$, $y_0 = zy$. Since yz is reduced, we may define $x_1 = x$, $y_1 = yz$, and observe that $(x_1,\ y_1)$ is the image of $(x_0,\ y_0)$ under conjugation by z , and $x_1,\ y_1$ are not coradical. We consider separately the cases in which x is or is not cyclically reduced.

Case 11.121. If x is not cyclically reduced, then no non-trivial cancellation can take place in $x \cdot y$, since $\bar{y}x$ is reduced. Therefore, by Corollary 11.1, we have

$$\lambda(W^*) \geq 2\big(\lambda(x)+\lambda(zy)\big) + N\{7\lambda(x)+10\lambda(zy)+3\}$$
$$\geq 4 + 12N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) .$$

Case 11.122. If x is cyclically reduced, we have

$$W^* = \cdot yz\bar{x})\{\bar{z}\bar{y}(x\bar{z}\bar{y}\cdot\bar{x}^2yz)yzx(x\bar{z}\bar{y}\cdot\bar{x}yz)\bar{x}\bar{z}\bar{y}\big(\bar{z}\bar{y}x^2\cdot(yz)^2\bar{x}\big)\}^N(\bar{z}\bar{y}x \; *$$

and by Theorem 13 we have

$$\lambda(W^*) \geq 4 + N\{4\lambda(x)+4\lambda(yz)+12\}$$
$$= 4 + 12N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) .$$

Case 11.13. The transcription $z \to \bar{x}z$ gives

$$W^* = \cdot zy\bar{x}\bar{z}x\{\bar{y}x\bar{y}\bar{z}\cdot\bar{z}\cdot y\bar{x}zy\bar{x}z\cdot zx\bar{y}\bar{z}\cdot y\bar{x}\bar{z}x\bar{y}\bar{z}x\bar{y}\cdot z\cdot zy\bar{x}zy\bar{x}\bar{z}x\}^{N}\bar{y}\cdot *$$

where $x_0 = \bar{x}zx$, $y_0 = \bar{x}zy$, $x \neq 1$, $y \neq 1$. We see that $y\bar{x}$ is reduced, and, by Theorem 7, z and $zy\bar{x}$ are not coradical. If we define $x_1 = z$, $y_1 = zy\bar{x}$, then (x_1, y_1) is the image of (x_0, y_0) under conjugation by $\bar{x}$.

We consider separately the cases in which z is or is not cyclically reduced.

Case 11.131. If z is not cyclically reduced, then no non-trivial cancellation can take place in $\bar{y}\cdot z$, since zy is reduced. Therefore, by Corollary 11.1 we have

$$\lambda(W^*) \geq 2\big(\lambda(x)+\lambda(y)+\lambda(z)\big) + N\{10\lambda(x)+10\lambda(y)+9\lambda(z)+3\}$$
$$\geq 6 + 16N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) ,$$

since $x \neq 1$, $y \neq 1$, and $z \neq 1$.

Case 11.132. If z is cyclically reduced, then the expression becomes

$$W^* = \cdot zy\bar{x})\{\bar{z}x\bar{y}(x\bar{y}\bar{z}^2\cdot y\bar{x}z)y\bar{x}z^2(x\bar{y}\bar{z}\cdot y\bar{x}\bar{z})x\bar{y}(\bar{z}x\bar{y}\cdot z^2y\bar{x})zy\bar{x}\}^N(\bar{z}x\bar{y}\cdot *$$

and, by Theorems 11 and 22, we have

$$\lambda(W^*) \geq 4 + N\big\{\lambda(\bar{z}x\bar{y})+\big(3\lambda(z)+4\big)+\lambda(y\bar{x}z^2)+\lambda(x\bar{y})+\big(\lambda(z)+4\big)+\lambda(zy\bar{x})\big\}$$
$$= 4 + 8N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) .$$

Case 11.2. If cancellation dies in $x_0\cdot y_0$, we transcribe $(x_0, y_0) \to (xz, \bar{z}y)$, where $x \neq 1$, $y \neq 1$, and $x\cdot y = xy$. This transcription gives

$$W^* = y\bar{z}\bar{x}\{\bar{y}z\cdot xzy\bar{x}\cdot\bar{z}\bar{x}\cdot\bar{z}y\bar{z}yxz\cdot xz\bar{y}\bar{x}\cdot\bar{z}y\bar{z}\bar{x}\bar{y}z\bar{y}z\cdot xz\cdot xy\bar{z}y\bar{z}\bar{x}\}^{N}\bar{y}z\cdot x^* .$$

We consider separately the cases in which cancellation in $z\cdot x$ dies, persists via the transcription $x \to \bar{z}x$, or persists via the transcription $z \to z\bar{x}$.

Case 11.21. If cancellation dies in $z\cdot x$, we transcribe $(x, z) \to (ux, z\bar{u})$ so as to give

$$W^* = yu\bar{z}\bar{x}\bar{u}\{\bar{y}zxz\bar{u}\bar{y}(\bar{x}\bar{z})^2yu\bar{z}yuxzxz\bar{u}\bar{y}\bar{x}\bar{z}yu\bar{z}\bar{x}\bar{u}\bar{y}z\bar{u}\bar{y}(zx)^2yu\bar{z}yu\bar{z}\bar{x}\bar{u}\}^{N}\bar{y}zx^*$$

where $x_0 = uxz\bar{u}$, $y_0 = u\bar{z}y$, $x \neq 1$, $z \neq 1$ and $y \neq 1$. Since yu is reduced, so is $\bar{z}yu$, and $x_1 = xz$, $y_1 = \bar{z}yu$ are the images of x_0, y_0 , respectively, under conjugation by u . The length of W^* is

$$2\big(\lambda(x)+\lambda(yu)+\lambda(z)\big) + N\{10\lambda(x)+10\lambda(yu)+14\lambda(z)\} \geq 6 + 18N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) .$$

Case 11.22. The transcription $x \to \bar{z}x$ gives

$$W^* = y\bar{z}\bar{x}z\{\bar{y}\cdot xz\bar{y}\bar{x}\cdot\bar{x}\cdot(y\bar{z})^2x\cdot xz\bar{y}\bar{x}\cdot y\bar{z}\bar{x}(z\bar{y})^2\cdot x\cdot x(y\bar{z})^2\bar{x}z\}^{N}\bar{y}\cdot x^*$$

where $x_0 = \bar{z}xz$, $y_0 = \bar{z}y$. Since $y\bar{z}$ is reduced, the image (x_1, y_1) of (x_0, y_0) under conjugation by $\bar{z}$ is given by $x_1 = x$, $y_1 = y\bar{z}$, and, by Theorem 7, x_1 and y_1 are not coradical. Now we ask whether x is cyclically reduced.

Case 11.221. If x is not cyclically reduced, then no non-trivial cancellation can take place in $\bar{y}\cdot x$, since xy is reduced and $y \neq 1$. The length of W^* is therefore at least

$$2\big(\lambda(x)+\lambda(y\bar{z})\big) + N\{7\lambda(x)+10\lambda(y\bar{z})+3\} \geq 4 + 12N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) .$$

Case 11.222. If x is cyclically reduced, we have

$$W^* = y\bar{z})\{\big(\bar{x}z\bar{y}\cdot xz\bar{y}\bar{x}^2\cdot(y\bar{z})^2x\big)x\big(z\bar{y}\bar{x}\cdot y\bar{z}\bar{x}(z\bar{y})^2\cdot x^2y\bar{z}\big)y\bar{z}\}^N(\bar{x}z\bar{y}\cdot x^*$$

and by Theorem 13 and Corollary 24.1 we have

$$\lambda(W^*) \geq 4 + 8N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) .$$

Case 11.23. The transcription $z \to z\bar{x}$ gives

$$W^* = \cdot yx\bar{z}\bar{x}\{\bar{y}z\cdot z\bar{x}\bar{y}\cdot\bar{z}\cdot\bar{z}yx\bar{z}yxz\cdot z\bar{x}\bar{y}\cdot\bar{z}yx\bar{z}\bar{x}\bar{y}z\bar{x}\bar{y}z\cdot z\cdot yx\bar{z}yx\bar{z}\bar{x}\}^{N}\bar{y}z\cdot{}^*$$

where $x_0 = xz\bar{x}$, $y_0 = x\bar{z}y$, and we have still $x \neq 1$, $y \neq 1$. Since yx is reduced, the image (x_1, y_1) of (x_0, y_0) under conjugation by x is given by $x_1 = z$, $y_1 = \bar{z}yx$, and by Theorem 7, x_1 and y_1 are not coradical. Now we ask whether z $(= x_1)$ is cyclically reduced.

Case 11.231. If z is not cyclically reduced, since $\bar{y}z$ is reduced we have $z\cdot y = zy$, and since $y \neq 1$ the length of W^* is at least

$$2\big(\lambda(x)+\lambda(\bar{z}y)\big) + N\{10\lambda(x)+10\lambda(y)+10\lambda(z)+4\} \geq 6 + 18N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) .$$

Case 11.232. If z is cyclically reduced we have

$$W^* = \cdot yx\bar{z})\{\bar{x}\bar{y}z(z\bar{x}\bar{y}\cdot\bar{z}^2yx)\bar{z}yxz(z\bar{x}\bar{y}\cdot\bar{z}yx)\bar{z}\bar{x}\bar{y}z(\bar{x}\bar{y}z^2\cdot yx\bar{z})yx\bar{z}\}^N(\bar{x}\bar{y}z\cdot *$$

and since z, $\bar{z}yx$ are not coradical neither are z, yx (by Corollary 4.1). Thus, by Theorem 13, the length of W^* is at least

$$4 + 12N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) .$$

This completes our discussion of Case 11.

Case 12. Non-trivial cancellation is restricted to the three contexts (A): $y_0\cdot x_0$, (B): $\bar{y}_0\cdot x_0$, and (F): $y_0\cdot y_0$. We have

$$W^* = y_0\bar{x}_0\cdot\{\cdot\bar{y}_0\cdot x_0\bar{y}_0\bar{x}_0{}^2\cdot y_0\cdot y_0\cdot x_0{}^2\bar{y}_0\bar{x}_0\cdot y_0\bar{x}_0\cdot\bar{y}_0\cdot\bar{y}_0\cdot x_0{}^2y_0\cdot y_0\bar{x}_0\cdot\}^N\cdot\bar{y}_0\cdot x_0{}^*$$

and, just as in Cases 10 and 11, we may assume that cancellation dies either in $y_0\cdot x_0$ or in $\bar{y}_0\cdot x_0$ (else we may transcribe $y_0 \to x_0y\bar{x}_0$, etc.).

Case 12.1. If cancellation dies in $y_0\cdot x_0$, we transcribe $(x_0, y_0) \to (\bar{z}x, yz)$, where $x \neq 1$, $y \neq 1$, and $y\cdot x = yx$. This transcription gives

$$W^* = yz\bar{x}\{\bar{y}\cdot\bar{z}x\bar{z}\bar{y}\bar{x}z\bar{x}z\cdot yz\cdot yx\bar{z}x\bar{z}\bar{y}\bar{x}z\cdot yz\bar{x}\bar{y}\cdot\bar{z}\bar{y}\cdot\bar{z}x\bar{z}xyz\cdot yz\bar{x}\}^N\bar{y}\cdot\bar{z}x^*$$

and, as usual, we consider whether cancellation in $z\cdot y$ dies, persists via $y \to \bar{z}y$, or persists via $z \to z\bar{y}$.

Case 12.11. If cancellation dies in $z\cdot y$ we transcribe $(y, z) \to (uy, z\bar{u})$ so as to give

$$W^* = uyz\bar{u}\bar{x}\{\bar{y}\bar{z}xu\bar{z}\bar{y}\bar{u}\bar{x}z\bar{u}\bar{x}(zy)^2(xu\bar{z})^2\bar{y}\bar{u}\bar{x}zyz\bar{u}\bar{x}(\bar{y}\bar{z})^2xu\bar{z}xuyzyz\bar{u}\bar{x}\}^N\bar{y}\bar{z}x^*$$

where $x_0 = u\bar{z}x$, $y_0 = uyz\bar{u}$, $y \neq 1$, $z \neq 1$, and we have still $x \neq 1$. The image of (x_0, y_0) under conjugation by u is (x_1, y_1) , defined by $x_1 = \bar{z}xu$, $y_1 = yz$, and the length of W^* is

$$2\big(\lambda(xu)+\lambda(y)+\lambda(z)\big) + N\{10\lambda(xu)+10\lambda(y)+12\lambda(z)\}$$
$$\geq 6 + 16N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) .$$

Case 12.12. The transcription $y \to \bar{z}y$ gives

$$W^* = \bar{z}yz\bar{x}\{\bar{y}\cdot xz\bar{y}(z\bar{x})^2\cdot y\cdot y(x\bar{z})^2\bar{y}z\bar{x}\cdot yz\bar{x}\bar{y}\cdot\bar{y}\cdot(x\bar{z})^2y\cdot yz\bar{x}\}^N\bar{y}\cdot x^*$$

where $x_0 = \bar{z}x$, $y_0 = \bar{z}yz$, and $x \neq 1$. The image of (x_0, y_0) under conjugation by $\bar{z}$ is (x_1, y_1) , given by $x_1 = x\bar{z}$, $y_1 = y$, and x_1, y_1 are not coradical, by Theorem 7.

Case 12.121. If y is not cyclically reduced, no non-trivial cancellation is possible in $\bar{x}\cdot y$ since $x \neq 1$ and yx is reduced. The length of W^* is therefore at least

$$2\big(\lambda(y)+\lambda(x\bar z)\big) + N\{7\lambda(y)+10\lambda(x\bar z)+3\} \geq 4 + 12N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) .$$

Case 12.122. If y is cyclically reduced, we have

$$W^* = \bar z y)\{\big(z\bar x\bar y\cdot x\bar z\bar y(z\bar x)^2\cdot y^2x\bar z\big)x\bar z\big(\bar y z\bar x\cdot yz\bar x\bar y^2\cdot(x\bar z)^2y\big)y\}^N(z\bar x\bar y\cdot x^*$$

and by Theorem 13 and Corollary 24.1 we have

$$\lambda(W^*) \geq 4 + 8N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) .$$

Case 12.13. The transcription $z \to z\bar y$ gives

$$W^* = yz\bar y\bar x\cdot\{\cdot\bar z xy\bar z\bar y\bar x z\bar y\bar x z\cdot z\cdot xy\bar z xy\bar z\bar y\bar x z\cdot z\bar y\bar x\cdot\bar z\cdot\bar z xy\bar z xyz\cdot z\bar y\bar x\cdot\}^N\cdot\bar z x^*$$

where $x_0 = y\bar z x$, $y_0 = yz\bar y$, $x \neq 1$, $y \neq 1$. The image of (x_0, y_0) under conjugation by y is (x_1, y_1) , given by $x_1 = \bar z xy$, $y_1 = z$. By Theorem 7, x_1 and y_1 are not coradical, and in particular $y_1 \neq 1$.

Case 12.131. If z is not cyclically reduced, no non-trivial cancellation can take place in $z\cdot x$, since $\bar x z$ is reduced and $\bar x \neq 1$. Thus we have

$$\begin{aligned}\lambda(W^*) &\geq 2\big(\lambda(xy)+\lambda(z)\big) + N\{10\lambda(xy)+10\lambda(z)+4\}\\ &\geq 6 + 18N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) .\end{aligned}$$

Case 12.132. If z is cyclically reduced, we have

$$W^* = y)\{(z\bar y\bar x\cdot\bar z xy)\bar z\bar y\bar x z(\bar y\bar x z^2\cdot xy\bar z)xy\bar z\bar y\bar x z(z\bar y\bar x\cdot\bar z^2xy)\bar z xyz\}^N(z\bar y\bar x\cdot\bar z x^*$$

and since $\bar z xy$, z are not coradical, by Corollary 4.1 neither are xy, z . Therefore by Theorem 13 we have

$$\lambda(W^*) \geq 4 + 12N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) .$$

Case 12.2. If cancellation dies in $y_0\cdot x_0$, we transcribe $(x_0, y_0) \to (zx, zy)$, where $x \neq 1$, $y \neq 1$, and $\bar y\cdot x = \bar y x$. This transcription gives

$$W^* = zy\bar x\bar z\cdot\{\cdot\bar y x\bar y\bar z\bar x\bar z\bar x y\cdot zy\cdot(zx)^2\bar y\bar z\bar x yx z\cdot\bar y\bar z\cdot\bar y xzxzy\cdot zy\bar x\bar z\cdot\}^N\cdot\bar y x^*$$

and we ask whether cancellation in $y\cdot z$ dies, persists via the transcription $y \to y\bar z$, or persists via the transcription $z \to \bar y z$.

Case 12.21. If cancellation dies in $y\cdot z$, we transcribe $(y, z) \to (y\bar u, uz)$ so as to give

$$W^* = uzy\bar u\bar x\bar z\{\bar y xu\bar y(\bar z\bar u\bar x)^2(yz)^2xuzxu\bar y\bar z\bar u\bar x yu\bar x(\bar z\bar y)^2(xuz)^2yzy\bar u\bar x\bar z\}^N\bar y z^*$$

where $x_0 = uzx\bar u$, $y_0 = uzy$, $x \neq 1$, $y \neq 1$, $z \neq 1$. The image (x_1, y_1) of (x_0, y_0) under conjugation by u is given by $x_1 = zx$, $y_1 = zyu$, and the length of W^* is at least

$$2\big(\lambda(xu)+\lambda(y)+\lambda(z)\big) + N\{10\lambda(xu)+10\lambda(y)+12\lambda(z)\}$$
$$\geq 6 + 16N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) .$$

Case 12.22. The transcription $y \to y\bar{z}$ gives

$$W^* = zy\bar{z}\bar{x}\cdot\{\cdot\bar{y}xz\bar{y}\bar{z}\bar{x}\bar{z}\bar{x}y\cdot y\cdot(xz)^2\bar{y}\bar{z}\bar{x}y\bar{z}\bar{x}\cdot\bar{y}\cdot\bar{y}xzxzy\cdot y\bar{z}\bar{x}\cdot\}^N\cdot\bar{y}x^*$$

where $x_0 = zx$, $y_0 = zy\bar{z}$, $x \neq 1$ as before and since xz is reduced we see that $x_1 = xz$, $y_1 = y$ defines (x_1, y_1) as the image of (x_0, y_0) under conjugation by z , and Theorem 7 shows that x_1, y_1 are not coradical.

Case 12.221. If y is not cyclically reduced, the presence of $\bar{x}y$ shows that $y\cdot x = yx$, so that W^* has length at least

$$2\big(\lambda(x)+\lambda(y)+\lambda(z)\big) + N\{10\lambda(x)+7\lambda(y)+10\lambda(z)+3\}$$
$$\geq 4 + 18N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) .$$

Case 12.222. If y is cyclically reduced, we have

$$W^* = z)(y\bar{z}\bar{x}\cdot\{\cdot\bar{y}xz)\bar{y}\bar{z}\bar{x}(\bar{z}\bar{x}y^2\cdot(xz)^2\bar{y})\bar{z}\bar{x}(y\bar{z}\bar{x}\cdot\bar{y}^2xz)xzy(y\bar{z}\bar{x}\cdot\}^{N}\bar{y}x^*$$

and by Theorem 13 the length of W^* is at least

$$4 + 12N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) .$$

Case 12.23. The transcription $z \to \bar{y}z$ gives

$$W^* = \bar{y}zy\bar{x}\bar{z}\cdot\{\cdot x\bar{y}\bar{z}y\bar{x}\bar{z}y\bar{x}\cdot z\cdot zx\bar{y}zx\bar{y}\bar{z}y\bar{x}y\bar{x}\bar{z}\cdot\bar{z}\cdot x\bar{y}zx\bar{y}z\cdot zy\bar{x}\bar{z}\cdot\}^N\cdot x^*$$

where $x_0 = \bar{y}zx$, $y_0 = \bar{y}zy$, $x \neq 1$ and $y \neq 1$. The image (x_1, y_1) of (x_0, y_0) under conjugation by $\bar{y}$ is given by $x_1 = zx\bar{y}$, $y_1 = z$, and Theorem 7 shows that x_1, y_1 are not coradical.

Case 12.231. If z is not cyclically reduced, then no non-trivial cancellation is possible in $\bar{x}\cdot z$, since zx is reduced. Therefore

$$\lambda(W^*) \geq 2\big(\lambda(x)+\lambda(y)+\lambda(z)\big) + N\{10\lambda(x)+10\lambda(y)+9\lambda(z)+3\}$$
$$\geq 6 + 16N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) .$$

Case 12.232. If z is cyclically reduced, we have

$$W^* = \bar{y}z)\{(y\bar{x}\bar{z}\cdot x\bar{y}\bar{z})y\bar{x}(\bar{z}y\bar{x}\cdot z^2x\bar{y})zx\bar{y}\bar{z}y\bar{x}(y\bar{x}\bar{z}^2\cdot x\bar{y}z)x\bar{y}z^2\}^N(y\bar{x}\bar{z}\cdot x^*$$

where, since x_1 and y_1 are not coradical, we have also $x\bar{y}$ and z not coradical, by Corollary 4.1. In estimating the length of the portion within the braces, we consider together the first bracketed portion and one of the others, so as to apply Theorem 23. With Theorem 13, this shows the length of W^* to be at least

$$4 + 8N + N\{3\lambda(z)+\lambda(y\bar{x})+\lambda(z)+2\lambda(x_1)+\lambda(z)+\lambda(x_1)+\lambda(y_1)\}$$
$$= 4 + 8N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) .$$

This completes our discussion of Case 12.

Case 13. Non-trivial cancellation is restricted to the three contexts (C): $x_0 \cdot y_0$, (D): $x_0 \cdot \bar{y}_0$, and (F): $y_0 \cdot y_0$. We have

$$W^* = \cdot y_0 \cdot \bar{x}_0 \{\bar{y}_0 x_0 \cdot \bar{y}_0 \cdot \bar{x}_0{}^2 y_0 \cdot y_0 x_0{}^2 \cdot \bar{y}_0 \cdot \bar{x}_0 y_0 \cdot \bar{x}_0 \bar{y}_0 \cdot \bar{y}_0 x_0{}^2 \cdot y_0 \cdot y_0 \cdot \bar{x}_0\}^N \bar{y}_0 x_0 \cdot *$$

and, by the same reasoning as in Cases 10, 11, and 12, we may assume that cancellation dies in at least one of the contexts $x_0 \cdot y_0$, $x_0 \cdot \bar{y}_0$.

Case 13.1. If cancellation dies in $x_0 \cdot y_0$, we transcribe $(x_0, y_0) \to (x\bar{z}, zy)$, where $x \neq 1$, $y \neq 1$, and $x \cdot y = xy$. This transcription gives

$$W^* = y \cdot z\bar{x}\{\bar{y}\bar{z}x\bar{z} \cdot \bar{y}\bar{x}z\bar{x}zy \cdot zyx\bar{z}x\bar{z} \cdot \bar{y}\bar{x}zy \cdot z\bar{x}\bar{y}\bar{z} \cdot \bar{y}\bar{z}x\bar{z}xy \cdot zy \cdot z\bar{x}\}^N \bar{y}\bar{z}x^*$$

and we consider separately whether cancellation in $y \cdot z$ dies, persists via the transcription $y \to y\bar{z}$, or persists via the transcription $z \to \bar{y}z$.

Case 13.11. If cancellation dies in $y \cdot z$, we transcribe $(y, z) \to (yu, \bar{u}z)$ so as to give

$$W^* = yz\bar{x}\{\bar{u}\bar{y}\bar{z}ux\bar{z}\bar{y}\bar{x}\bar{u}z\bar{x}\bar{u}zyzyux\bar{z}ux\bar{z}\bar{y}\bar{x}\bar{u}zyz\bar{x}\bar{u}\bar{y}\bar{z}\bar{y}\bar{z}ux\bar{z}ux(yz)^2\bar{x}\}^N \bar{u}\bar{y}\bar{z}ux^*$$

where $x_0 = x\bar{z}u$, $y_0 = \bar{u}zyu$, $x \neq 1$, $y \neq 1$, and $z \neq 1$. The image (x_1, y_1) of (x_0, y_0) under conjugation by $\bar{u}$ is given by $x_1 = ux\bar{z}$, $y_1 = zy$, since ux is reduced. Thus we have

$$\begin{aligned} \lambda(W^*) &= 2\big(\lambda(ux)+\lambda(y)+\lambda(z)\big) + N\{10\lambda(ux)+10\lambda(y)+14\lambda(z)\} \\ &\geq 6 + 18N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) . \end{aligned}$$

Case 13.12. The transcription $y \to y\bar{z}$ gives

$$W^* = y \cdot \bar{x}\{z\bar{y}\bar{z}x \cdot \bar{y}(\bar{x}z)^2 y \cdot y(\bar{z}x)^2 \cdot \bar{y}\bar{x}zy \cdot \bar{x}z\bar{y} \cdot \bar{y}(\bar{z}x)^2 y \cdot y \cdot \bar{x}\}^N z\bar{y}\bar{z}x^*$$

where $x_0 = x\bar{z}$, $y_0 = zy\bar{z}$, and $x \neq 1$. Since $\bar{z}x$ is reduced, the image of (x_0, y_0) under conjugation by z is (x_1, y_1) , given by $x_1 = \bar{z}x$, $y_1 = y$, and by Theorem 7, x_1 and y_1 are not coradical.

Case 13.121. If y is not cyclically reduced, then no non-trivial cancellation is possible in $y \cdot \bar{x}$, since xy is reduced. Therefore

$$\begin{aligned} \lambda(W^*) &\geq 2\big(\lambda(\bar{z}x)+\lambda(y)\big) + N\{10\lambda(\bar{z}x)+7\lambda(y)+3\} \\ &\geq 4 + 12N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) . \end{aligned}$$

Case 13.122. If y is cyclically reduced, we have

$$W^* = y \cdot \bar{x}\{z\bar{y}\bar{z}x \cdot \bar{y}\bar{x}z)\bar{x}zy\big(y(\bar{z}x)^2 \cdot \bar{y}\bar{x}zy \cdot \bar{x}z\bar{y}\big)\bar{y}\bar{z}x(\bar{z}xy^2 \cdot \bar{x}\}^N z\bar{y})(\bar{z}x^*$$

and by Theorem 13 and Corollary 24.1 the length of W^* is at least

$$\lambda(y) + 4 + (N-1)\left(2\lambda(y)+4\right) + \lambda(y) + 4 + N\{2\lambda(\bar{x}zy)+2\lambda(\bar{z}x)+4\}$$
$$= 4 + 8N + 4N\left(\lambda(x_1)+\lambda(y_1)\right) .$$

Case 13.13. The transcription $z \to \bar{y}z$ gives

$$W^* = \cdot z\bar{x}\{\bar{y}\bar{z}yx\bar{z}\cdot\bar{x}\bar{y}z\bar{x}\bar{y}z\cdot zyx\bar{z}yx\bar{z}\cdot\bar{x}\bar{y}z\cdot z\bar{x}\bar{y}\bar{z}\cdot\bar{z}yx\bar{z}yx\cdot z\cdot z\bar{x}\}^N\bar{y}\bar{z}yx\cdot *$$

where $x_0 = x\bar{z}y$, $y_0 = \bar{y}zy$, $x \neq 1$, $y \neq 1$. The image of (x_0, y_0) under conjugation by $\bar{y}$ is (x_1, y_1) , given by $x_1 = yx\bar{z}$, $y_1 = z$. By Theorem 7, x_1 and y_1 are not coradical, hence $z \neq 1$.

Case 13.131. If z is not cyclically reduced, then no non-trivial cancellation can take place in $x\cdot z$, since $z\bar{x}$ appears as a reduced word. Therefore the length of W^* is at least

$$2\left(\lambda(x)+\lambda(y)+\lambda(z)\right) + N\{10\lambda(x)+10\lambda(y)+10\lambda(z)+4\}$$
$$\geq 6 + 18N + 4N\left(\lambda(x_1)+\lambda(y_1)\right) .$$

Case 13.132. If z is cyclically reduced, we have

$$W^* = \cdot z\bar{x}\bar{y})\{\bar{z}(yx\bar{z}\cdot\bar{x}\bar{y}z)\bar{x}\bar{y}z^2yx\bar{z}(yx\bar{z}\cdot\bar{x}\bar{y}z)z\bar{x}\bar{y}\bar{z}^2yx(\bar{z}yx\cdot z^2\bar{x}\bar{y})\}^N(\bar{z}yx\cdot *$$

and since $x_1 = yx\bar{z}$ and $y_1 = z$ are not coradical, neither, by Corollary 4.1, are yx and z . Therefore, by Theorem 13,

$$\lambda(W^*) \geq 4 + 12N + 4N\left(\lambda(x_1)+\lambda(y_1)\right) .$$

Case 13.2. If cancellation dies in $x_0\cdot\bar{y}_0$, we transcribe $(x_0, y_0) \to (xz, yz)$, where $x \neq 1$, $y \neq 1$, and $x\cdot\bar{y} = x\bar{y}$. This transcription gives

$$W^* = \cdot y\bar{x}\{\bar{z}\bar{y}x\bar{y}\cdot\bar{z}\bar{x}\bar{z}\bar{x}yz\cdot yzxzx\bar{y}\cdot\bar{z}\bar{x}y\bar{x}\bar{z}\bar{y}\cdot\bar{z}\bar{y}xzxz\cdot yz\cdot y\bar{x}\}^N\bar{z}\bar{y}xz\cdot *$$

and we consider separately whether cancellation in $x\cdot y$ dies, persists via the transcription $y \to \bar{z}y$, or persists via the transcription $z \to z\bar{y}$.

Case 13.21. If cancellation dies in $z\cdot y$, we transcribe $(y, z) \to (\bar{u}y, zu)$ so as to give

$$W^* = y\bar{x}\{\bar{u}\bar{z}\bar{y}ux\bar{y}\bar{z}\bar{x}\bar{u}\bar{z}\bar{x}\bar{u}\bar{y}zy(zux)^2\bar{y}\bar{z}\bar{x}\bar{u}y\bar{x}\bar{u}\bar{z}\bar{y}\bar{z}\bar{y}uxzux(zy)^2\bar{x}\}^N\bar{u}\bar{z}\bar{y}uxz *$$

where $x_0 = xzu$, $y_0 = \bar{u}yzu$, $x \neq 1$, $y \neq 1$, $z \neq 1$. The image (x_1, y_1) of (x_0, y_0) under conjugation by $\bar{u}$ is given by $x_1 = uxz$, $y_1 = yz$, since ux is reduced. Direct computation gives

$$\lambda(W^*) = 2\left(\lambda(ux)+\lambda(y)+\lambda(z)\right) + N\{10\lambda(ux)+10\lambda(y)+12\lambda(z)\}$$
$$\geq 6 + 16N + 4N\left(\lambda(x_1)+\lambda(y_1)\right) .$$

Case 13.22. The transcription $y \to \bar{z}y$ gives

$$W^* = \cdot y\bar{x}\{\bar{z}\bar{y}zx\bar{y}\cdot(\bar{x}\bar{z})^2y\cdot y(zx)^2\bar{y}\cdot\bar{x}\bar{z}y\bar{x}\bar{z}\bar{y}\cdot\bar{y}(zx)^2\cdot y\cdot y\bar{x}\}^N\bar{z}\bar{y}zx\cdot *$$

where $x_0 = xz$, $y_0 = \bar{z}yz$, $x \neq 1$, $y \neq 1$. The image of (x_0, y_0) under conjugation by $\bar{z}$ is (x_1, y_1) , given by $x_1 = zx$, $y_1 = y$, and by Theorem 7, x_1 and y_1 are not coradical, hence $y \neq 1$.

Case 13.221. If y is not cyclically reduced, then no non-trivial cancellation is possible in $x\cdot y$, since $y\bar{x}$ is reduced. Therefore

$$\begin{aligned}\lambda(W^*) &\geq 2\big(\lambda(zx)+\lambda(y)\big) + N\{10\lambda(zx)+7\lambda(y)+3\} \\ &\geq 4 + 12N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) .\end{aligned}$$

Case 13.222. If y is cyclically reduced, we have

$$W^* = \cdot y\bar{x}\bar{z})\{\bar{y}\big(zx\bar{y}\cdot(\bar{x}\bar{z})^2y\big)yzx(zx\bar{y}\cdot\bar{x}\bar{z}y)\bar{x}\bar{z}\bar{y}\big(\bar{y}(zx)^2\cdot y^2\bar{x}\bar{z}\big)\}^N(\bar{y}zx\cdot *$$

and by Theorem 13 we have

$$\lambda(W^*) \geq 4 + 12N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) .$$

Case 13.23. The transcription $z \to z\bar{y}$ gives

$$W^* = \cdot\bar{x}\{y\bar{z}\bar{y}x\cdot\bar{z}\bar{x}y\bar{z}\bar{x}yz\cdot z\bar{y}xz\bar{y}x\cdot\bar{z}\bar{x}y\bar{x}y\bar{z}\cdot\bar{z}\bar{y}xz\bar{y}xz\cdot z\cdot\bar{x}\}^N y\bar{z}\bar{y}xz\cdot *$$

where $x_0 = xz\bar{y}$, $y_0 = yz\bar{y}$, $x \neq 1$, $y \neq 1$. The image of (x_0, y_0) under conjugation by y is (x_1, y_1) , where $x_1 = \bar{y}xz$, $y_1 = z$. By Theorem 7, x_1 and y_1 are not coradical, so in particular $y_1 \neq 1$.

Case 13.231. If z is not cyclically reduced, then no non-trivial cancellation can take place in $z\cdot\bar{x}$, since xz is reduced. Therefore

$$\begin{aligned}\lambda(W^*) &\geq 2\big(\lambda(x)+\lambda(y)+\lambda(z)\big) + N\{10\lambda(x)+10\lambda(y)+9\lambda(z)+3\} \\ &\geq 6 + 16N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) .\end{aligned}$$

Case 13.232. If z is cyclically reduced, we have

$$W^* = \cdot\bar{x}\{y\bar{z}\bar{y}x\cdot\bar{z}\bar{x}y)\bar{z}\bar{x}yz^2\bar{y}x(z\bar{y}x\cdot\bar{z}\bar{x}y)\bar{x}y\bar{z}^2\bar{y}xz(\bar{y}xz^2\cdot\bar{x}\}^N y\bar{z})(\bar{y}xz\cdot *$$

where, since x_1 and y_1 are not coradical, by Corollary 4.1 neither are $\bar{y}x$ and z . But now Theorem 13 and Corollary 24.1 show that the length of W^* is at least

$$\begin{aligned}\lambda(z) + 4 + (N-1)\big(2\lambda(z)+4\big) + \lambda(z) + 4 + N\{2\lambda(x_1)+\lambda(y_1)+4+2\lambda(x_1)+\lambda(y_1)\} \\ = 4 + 8N + 4N\big(\lambda(x_1)+\lambda(y_1)\big) .\end{aligned}$$

This completes our discussion of Case 13.

If non-trivial cancellation is possible in all of the contexts (A)-(F) between x_0, y_0, $\bar{x}_0$ and $\bar{y}_0$, then by Corollary 28.1 there are words x_1, y_1, u in F, such that $x_0 = ux_1\bar{u}$, $y_0 = uy_1\bar{u}$, and non-trivial cancellation is ruled out in at least one of the contexts $y_1 \cdot x_1$, $\bar{y}_1 \cdot x_1$, $x_1 \cdot y_1$, $x_1 \cdot \bar{y}_1$, $x_1 \cdot x_1$, and $y_1 \cdot y_1$. But $W(x_0, y_0)^* = W(x_1, y_1)^*$, and now if we multiply out the expression for $W(x_1, y_1)^*$ and consider cases just as we have done for $W(x_0, y_0)^*$, we find that cancellation in $W(x_1, y_1)^*$ falls under Case 3 or one of Case 8, ..., Case 13, so that the length of $W(x_1, y_1)^*$ is at least $4 + 8N + 4N\left(\lambda(x_1^v)+\lambda(y_1^v)\right)$ for some word v, hence $\lambda\left(W(x_0, y_0)^*\right) \geq 4 + 8N + 4N\left(\lambda(x_0^z)+\lambda(y_0^z)\right)$, where $z = u \cdot v$. This completes the proof of the Theorem. //

To verify that the bound given by the above theorem is sharp, we inspect one of the sub-cases for which we arrived at precisely this bound in order to find suitable values to assign to x_0 and y_0. We choose Case 9.311, in which we had $x_0 = (xy)^m x$, $y_0 = (xy)^n x$, where x, y were not coradical, and m, n were positive integers, with $m = n + 1$. We computed the length of W^* as

$$\lambda(yx \cdot \bar{y}\bar{x} \cdot yx \cdot \bar{y}\bar{x}) - \lambda(yx \cdot \bar{y}\bar{x}) + N\left\{8\lambda\left((xy)^n x\right)+2\lambda(xy \cdot \bar{x}\bar{y})+2\lambda(\bar{x}\bar{y} \cdot xy)\right\}$$
$$\geq 4 + N\left\{8\lambda\left((xy)^n x\right)+4\lambda(xy)+8\right\} .$$

Equality holds if x, y are mutually independent letters, so that in this case the length of $W(x_0, y_0)^*$ is precisely $4 + 8N + 4N\left(\lambda(x_0)+\lambda(y_0)\right)$. Since the value of n is unspecified, this bound remains sharp even if we place lower limits on $\lambda(x_0{}^*)$ and $\lambda(y_0{}^*)$. If instead (or in addition) we insist that $\left|\lambda(x_0{}^*)-\lambda(y_0{}^*)\right|$ should be large, we can find suitable examples following Case 9.31231 or Case 9.32231, in which k can be made arbitrarily large.

By inspection of the above proof, or by comparing the result just proved with the corresponding result for $N + 1$ in place of N, we have

COROLLARY 29.1. *If x, y are arbitrary words in F such that $x \cdot y \neq y \cdot x$, then the cyclically reduced length of*

$$\left[[x \cdot x,\ y \cdot y],\ [x,\ y]\right]$$

is at least equal to

$$8 + 4\left(\lambda(x^z)+\lambda(y^z)\right)$$

for some word z in F, depending on x and y. //

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